Post-Newtonian Gravitational Radiation Reaction for Two-Body Systems

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We study gravitational radiation reaction in the equations of motion for binary systems to post-Newtonian order $[O(v/c)^2]$ beyond the quadrupole approximation. The method uses post-Newtonian expressions for energy and angular momentum flux to infinity, and an assumption of energy and angular momentum balance. The equations of motion are valid for general binary orbits. We discuss the coordinate-system dependence of the radiation-reaction formula.

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During the past twenty years, gravitational-radiation damping has been recognized as a process with important observational consequences. Eighteen years of observation of the binary pulsar 1913+16 have yielded a verification of the "quadrupole formula" for radiation damping to a precision of better than half a percent [1]. Laser interferometric gravitational-wave observatories are expected to have the capability to detect waves from the final inspiral and coalescence of two compact objects (neutron stars or black holes), a process dominated by gravitational-radiation damping. Indeed, because of the broadband nature of such detectors, it will be possible to study the gravitational wave form as a function of time, and thereby to determine important parameters of the source, such as the masses, spins, and radii of the two bodies [2, 3].

Because an inspiralling binary system will emit a gravitational-wave signal with the characteristics of a "chirp" (amplitude and frequency increasing with time), the data analysis process involves cross correlating the data with a set of theoretical signal templates for coalescing binaries with a range of system parameters. To this end, it is important that the templates be as accurate as possible. Approximate orbital evolutions and gravitational wave forms have been calculated using high-order post-Newtonian expansions [4–6], and, ultimately, full-scale numerical relativity computer codes will play a role [7]. Recently, Cutler *et al.* [3] have emphasized the importance of knowing the secular damping of the orbit very accurately, in order that the theoretical template not lose phase with the observed signal during the period when the orbital frequency sweeps through the detector's bandwidth, typically between 10 and 1000 Hz.

This Letter addresses the question of the accuracy of gravitational-radiation damping, and presents, for the first time, a formula suitable for determining the evolution of general inspiralling binary orbits, that includes the first post-Newtonian $[O(v/c)^2]$ corrections to the dominant Newtonian, or quadrupole radiation-damping terms.

In the usual post-Newtonian expansion of the equations of motion in terms of a small parameter $\epsilon \sim (v/c)^2 \sim Gm/rc^2$, where m, v, and r are the total mass, orbital velocity, and separation of the binary system, gravitational radiation reaction first appears at $O(\epsilon^{5/2})$ beyond Newtonian gravitation. We call this "Newtonian radiation reaction." Here we obtain the terms at $O(\epsilon^{7/2})$, or post-Newtonian radiation reaction. In this paper, we outline the method and present the results; details will be published elsewhere [8].

Numerous authors have obtained, from first principles, approximate solutions of Einstein's equations that incorporate the backreaction from radiation to infinity into "near-zone" gravitational fields, leading to the Newtonian radiation-reaction terms in the equations of motion [9]. To date, however, such calculations have not been extended any higher, apart from specific effects, such as those due to gravitational-wave tails [10].

On the other hand, post-Newtonian corrections to the far-zone gravitational wave form for binary systems *have* been calculated through $O(\epsilon^{3/2})$ beyond the quadrupole formula [5, 11–13], resulting, for example, in formulas for the flux of energy, angular momentum, and linear momentum valid to post-Newtonian order, i.e., $O(\epsilon)$ beyond the quadrupole formula. The reason for this dichotomy is that calculation of far-zone gravitational waves to a given accuracy in a post-Newtonian expansion requires one fewer iteration of the "relaxed Einstein equations" [schematically $\Box h_{\mu\nu} \approx T_{\mu\nu} + (\partial_\alpha h_{\beta\gamma})^2$] than does calculation of near-zone radiation reaction [14], and hence is "easier." Our method exploits these post-Newtonian energy and angular momentum flux formulas to "derive" a radiation-reaction formula that guarantees a compensating loss in the source. The surprise is that we obtain a formula that is applicable for arbitrary two-body orbits (within the weak-field slow-motion constraints of the post-Newtonian method), and that is unique in a sense to be described below (for an elementary Newtonian version of this argument, see [15]).

Specifically, the method proceeds as follows: We write down a general form for the Newtonian ($\epsilon^{5/2}$) and post-Newtonian ($\epsilon^{7/2}$) radiation-reaction terms in the equations of motion for two bodies, ignoring tidal and spin effects. For the relative acceleration $\mathbf{a} \equiv \mathbf{a}_2 - \mathbf{a}_1$, this has the form ($G = c = 1$)

$$\mathbf{a} = -\frac{8}{5}\eta(m/r^2)(m/r) \times \left[-(A_{5/2} + A_{7/2})\dot{r}\mathbf{n} + (B_{5/2} + B_{7/2})\mathbf{v}\right], \quad (1)$$

where $m \equiv m_1 + m_2$ is the total mass; $\mu \equiv m_1 m_2/m$ is the reduced mass, with $\eta = \mu/m$; $\mathbf{x}$, r, and $\mathbf{v}$ are the relative separation vector, distance, and velocity between the two bodies, with $\mathbf{n} = \mathbf{x}/r$. The form of Eq. (1) is dictated by the fact that it must be a correction to the Newtonian acceleration (m/r^2), vanish in the test body limit (η), be related to the emission of gravitational radiation (another m/r), and be dissipative, or odd in velocities ($\dot{r}\mathbf{n}$, $\mathbf{v}$). The prefactor 8/5 is chosen for convenience. Then to make the leading term of $O(\epsilon^{5/2})$ beyond Newtonian order, $A_{5/2}$ and $B_{5/2}$ must be of $O(\epsilon)$. The only variables in the problem of this order are v^2, m/r, and $\dot{r}^2$. Thus $A_{5/2}$ and $B_{5/2}$ each can consist of a linear combination of these three terms; to those terms we assign six arbitrary parameters. By the same reasoning, $A_{7/2}$ and $B_{7/2}$ must be of $O(\epsilon^2)$; hence each must be a linear combination of the six terms v^4, $v^2 m/r$, $v^2\dot{r}^2$, $\dot{r}^2 m/r$, $\dot{r}^4$, and $(m/r)^2$. To these we assign twelve arbitrary parameters. We ignore terms in the equations of motion of $O(\epsilon^3)$ beyond Newtonian order, because they are nondissipative [there is a clean split between integer order and odd-half-integer order in this procedure, at least through $O(\epsilon^{7/2})$].

We take post-post-Newtonian expressions for orbital energy and angular momentum (per unit reduced mass), $\tilde{E} \equiv E/\mu = \frac{1}{2}v^2 - m/r + O(\epsilon^2) + O(\epsilon^3)$, $\tilde{\mathbf{J}} = \mathbf{x} \times \mathbf{v}[1 + O(\epsilon) + O(\epsilon^2)]$ [6, 16], and calculate $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$ using post-post-Newtonian two-body equations of motion [4] supplemented by the dissipative terms of Eq. (1). Through post-post-Newtonian order, $\tilde{E}$ and $\tilde{\mathbf{J}}$ are constant, and correspond to asymptotically measured quantities, but the radiation-reaction terms lead to nonvanishing expressions for $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$. However, we have the freedom to add to $\tilde{E}$ and $\tilde{\mathbf{J}}$ arbitrary terms of order $\epsilon^{5/2}$ and $\epsilon^{7/2}$ beyond the Newtonian expressions without affecting their conservation at post-post-Newtonian order. An example would be a term $\frac{8}{5}\eta(m/r)^2 v^2 \dot{r}$ in $\tilde{E}$ [17]. There are six such terms at $O(\epsilon^{5/2})$ in $\tilde{E}$ and $\tilde{\mathbf{J}}$ and twelve at $O(\epsilon^{7/2})$. Adding arbitrary amounts of each such term to $\tilde{E}$ and $\tilde{\mathbf{J}}$ has the effect of changing the form of the residual terms in $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$ by terms involving six additional free parameters at $O(\epsilon^{5/2})$ and twelve at $O(\epsilon^{7/2})$.

We now equate the residual $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$ expressions to the negatives of the corresponding far-zone flux formulas, and compare them term by term. (Although this assumption of energy and angular momentum balance is eminently reasonable, it has not been justified rigorously to date.) For example, the Newtonian energy and angular momentum flux formulas have the forms

$$\dot{\tilde{E}}_{\text{far zone}} = \frac{8}{5}\eta(m/r^2)(m/r)\left[4v^2 m/r - \frac{11}{3}\dot{r}^2 m/r\right], \quad (2a)$$

$$\dot{\tilde{\mathbf{J}}}_{\text{far zone}} = \frac{8}{5}\eta(m/r^2)\tilde{\mathbf{J}}\left[2v^2 m/r - 3\dot{r}^2 m/r + 2(m/r)^2\right]. \quad (2b)$$

Since the terms in square brackets could also have included v^4, $v^2\dot{r}^2$, and $\dot{r}^4$, setting $d\tilde{E}/dt = -\dot{\tilde{E}}_{\text{far zone}}$ and $d\tilde{\mathbf{J}}/dt = -\dot{\tilde{\mathbf{J}}}_{\text{far zone}}$ yields a total of twelve constraints on the coefficients at Newtonian ($\epsilon^{5/2}$) order. At post-Newtonian order, the corresponding expressions involve v^6, $v^4\dot{r}^2$, etc., a total of ten terms each in $\dot{\tilde{E}}_{\text{far zone}}$ and $\dot{\tilde{\mathbf{J}}}_{\text{far zone}}$, resulting in twenty constraints on the post-Newtonian ($\epsilon^{7/2}$) coefficients [18].

Of the twelve constraints at Newtonian order, two are not linearly independent, resulting in ten constraints on the twelve parameters. Solving these constraints results in the form

$$A_{5/2} = 3(1+\beta)v^2 + \frac{1}{3}(23 + 6\alpha - 9\beta)m/r - 5\beta\dot{r}^2, \quad (3a)$$

$$B_{5/2} = (2+\alpha)v^2 + (2-\alpha)m/r - 3(1+\alpha)\dot{r}^2, \quad (3b)$$

where α and β represent the remaining 2 unconstrained degrees of freedom. The choice $\alpha = -1$, $\beta = 0$ leads to the Damour-Deruelle two-body radiation-reaction formula used in [4]; the choice $\alpha = 4$, $\beta = 5$ leads to the form obtained from the "Burke-Thorne" radiation reaction potential $\Phi_{\text{RR}} = \frac{1}{5}d^5 Q_{ij}/dt^5 x^i x^j$, where Q_{ij} is the trace-free moment-of-inertia tensor of the system [19]. In fact, it is straightforward to show that the arbitrariness represented by α and β is a consequence of the freedom to make coordinate transformations whose resultant effect on the two-body separation vector is $\mathbf{x} \to \mathbf{x} + \frac{8}{15}\eta(m/r)^2[\beta\dot{r}\mathbf{x} + (2\beta - 3\alpha)r\mathbf{v}]$, or $\delta r/r = \frac{8}{5}\eta(m/r)^2\dot{r}(\beta - \alpha)$. The 2 degrees of freedom correspond to the possible functional forms of such transformations at $O(\epsilon^{5/2})$. In a binary coalescence of equal-mass compact objects, for example, this will change the coordinate separation by only 2 parts in 10^7 at a separation $r = 20m$, and 3 parts in 10^4 at the innermost stable orbit at $r = 6m$, for values of α and β of order unity. Similarly, the coordinate transformation yields $\delta\dot{\phi}/\dot{\phi} = -\frac{24}{5}\eta(m/r)^2\dot{r}(2\beta - 3\alpha)$, from which it is simple to show that the accumulated correction in the orbital phase during the coalescence will amount to only 5×10^{-3} rad. Consequently, one can choose α and β freely; the error made by using coordinate variables instead of invariant quantities is negligible for systems of interest.

At post-Newtonian order, of the twenty constraints, two are again linearly dependent on the others, resulting in eighteen constraints on 24 parameters. Solving these constraints results in

$$A_{7/2} = a_1 v^4 + a_2 v^2 m/r + a_3 v^2 \dot{r}^2 + a_4 \dot{r}^2 m/r + a_5 \dot{r}^4 + a_6 (m/r)^2 , \tag{4a}$$

$$B_{7/2} = b_1 v^4 + b_2 v^2 m/r + b_3 v^2 \dot{r}^2 + b_4 \dot{r}^2 m/r + b_5 \dot{r}^4 + b_6 (m/r)^2 , \tag{4b}$$

where the twelve coefficients are given by

$$a_1 = \frac{1}{28}(117 + 132\eta) - \frac{3}{2}\beta(1 - 3\eta) + 3\delta_2 - 3\delta_6 , \tag{5a}$$

$$a_2 = -\frac{1}{42}(297 - 310\eta) - 3\alpha(1 - 4\eta) - \frac{3}{2}\beta(7 + 13\eta) - 2\delta_1 - 3\delta_2 + 3\delta_5 + 3\delta_6 , \tag{5b}$$

$$a_3 = \frac{5}{28}(19 - 72\eta) + \frac{5}{2}\beta(1 - 3\eta) - 5\delta_2 + 5\delta_4 + 5\delta_6 , \tag{5c}$$

$$a_4 = -\frac{1}{28}(687 - 368\eta) - 6\alpha\eta + \frac{1}{2}\beta(54 + 17\eta) - 2\delta_2 - 5\delta_4 - 6\delta_5 , \tag{5d}$$

$$a_5 = -7\delta_4 , \tag{5e}$$

$$a_6 = -\frac{1}{21}(1533 + 498\eta) - \alpha(14 + 9\eta) + 3\beta(7 + 4\eta) - 2\delta_3 - 3\delta_5 , \tag{5f}$$

$$b_1 = -3(1 - 3\eta) - \frac{3}{2}\alpha(1 - 3\eta) - \delta_1 , \tag{6a}$$

$$b_2 = -\frac{1}{84}(139 + 768\eta) - \frac{1}{2}\alpha(5 + 17\eta) + \delta_1 - \delta_3 , \tag{6b}$$

$$b_3 = \frac{1}{28}(369 - 624\eta) + \frac{3}{2}(3\alpha + 2\beta)(1 - 3\eta) + 3\delta_1 - 3\delta_6 , \tag{6c}$$

$$b_4 = \frac{1}{42}(295 - 335\eta) + \frac{1}{2}\alpha(38 - 11\eta) - 3\beta(1 - 3\eta) + 2\delta_1 + 4\delta_3 + 3\delta_6 , \tag{6d}$$

$$b_5 = \frac{5}{28}(19 - 72\eta) - 5\beta(1 - 3\eta) + 5\delta_6 , \tag{6e}$$

$$b_6 = -\frac{1}{21}(634 - 66\eta) + \alpha(7 + 3\eta) + \delta_3 . \tag{6f}$$

The 6 unconstrained degrees of freedom are represented by $\delta_1, \ldots, \delta_6$; notice also that the parameters α and β appear as a result of post-Newtonian corrections to the Newtonian terms. It can again be shown that the $(\alpha, \beta, \delta_i)$ freedom is a consequence of coordinate transformations, with $\delta\mathbf{x} \sim \frac{8}{5}\eta(m/r)^2 [O(\epsilon)\dot{r}\mathbf{x} + O(\epsilon)\mathbf{v}]$, with the 6 degrees of freedom here corresponding to the six possible $O(\epsilon)$ terms in $\delta\mathbf{x}$. The δ parameters could be chosen, for instance, so as to eliminate six of the twelve terms in $A_{7/2}$ and $B_{7/2}$. These formulas can be easily implemented for evolving coalescing binary orbits. For nearly circular, inspiralling orbits, the orbit equations of [4] yield

$$\dot{r} = -\frac{64}{5}\eta\left(\frac{m}{r}\right)^3\left[1 - \left(\frac{1751}{336} + \frac{7}{4}\eta\right)\frac{m}{r}\right] , \tag{7}$$

and for the orbital angular frequency,

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\eta(m\omega)^{5/3}\left[1 - \left(\frac{743}{336} + \frac{11}{4}\eta\right)\left(m\omega\right)^{2/3}\right] , \tag{8}$$

in agreement with [13] and [3] to the corresponding order.

This general approach can also be extended to derive radiation-reaction expressions caused by gravitational-wave tail effects [5, 10], which occur at $O(\epsilon^4)$, spin effects [20], and possibly higher-order effects, once the appropriate flux formulas have been derived. We also remark that, by focusing on the relative equations of motion, we are ignoring the effects of a net radiation of linear momentum by the system [5]. This radiation will correspond to radiation-reaction terms in the individual accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ that will lead to a net center-of-mass acceleration, $\mathbf{a}_{\mathrm{c.m.}} = (m_1\mathbf{a}_1 + m_2\mathbf{a}_2)/m$. However, we believe that this will not result in further constraints on the coefficients of the relative acceleration beyond those derived above. This question is currently under study.

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