Magnetization of all stationary cylindrically symmetric vacuum metrics

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New exact solutions of the Einstein–Maxwell equations are obtained by a Harrison-type transformation applied to the most general stationary cylindrically symmetric vacuum spacetime. These solutions include generalizations of the well-known Melvin Magnetic Universe, and the Weyl–Levi-Cività magnetic solution in addition to a new class of magnetic stationary axisymmetric spacetimes.

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I. INTRODUCTION

The importance of magnetic fields in astrophysical situations is well recognized and one convenient way to introduce a magnetic field in some given spacetimes is by a Harrison[1]-type transformation. Ernst[2] used this method to obtain the solution corresponding to a black hole in an external magnetic field. Though the method leads to nonasymptotically flat spacetimes and sourceless electromagnetic fields the Ernst solution has been useful and employed as an idealized model by various authors[3] to study the effect of magnetic fields in different physical situations. However, especially in the Kerr[4] case, the forms of the fields are very complicated and it is difficult to understand their structure. Consequently it would be instructive if one could study the electromagnetic fields that arise from a Harrison-type transformation in a case where the form of the functions involved is simpler.

To initiate such an investigation we apply a Harrison-type transformation to the stationary cylindrically symmetric vacuum metrics because the most general form of these solutions has been given in an especially simple form by Vishveshwara and Winicour.[5] Further, these solutions include as a special case the flat Minkowski metric so that we expect to obtain similar but more general spacetimes than the Melvin magnetic Universe (MMU).[6] Indeed as we shall see later the solutions include generalizations of the twisted MMU (TMMU)[1] and the twisted Weyl–Levi-Cività magnetic solution (TWLCM). In addition a new class of magnetic stationary metrics is generated which is axisymmetric and no longer retain their original cylindrical symmetry.

Recently Hiscock[7] has pointed out that the magnetized Kerr black hole is asymptotically non-Melvin unlike the magnetized Schwarzschild black hole although both the metrics in their original nonmagnetized versions are asymptotically flat. This follows by noting that in the magnetic Kerr case the electric fields persist even in the asymptotic region. It would be of obvious interest to check whether asymptotically the magnetic Kerr solution goes into any one of the above new classes of solutions. These above three reasons motivate us to obtain this new class of magnetized solutions starting from the most general stationary cylindrically symmetric vacuum spacetime.

In the next section, the Harrison transformation used by Ernst[2] to obtain magnetized solutions is given. Next following Vishveshwara and Winicour[5] the most general stationary cylindrically symmetric vacuum metric is written down and its relevant properties summarized for later use. The Harrison transformation is then applied and it is shown that the equation for ω' can be explicitly integrated to obtain the new magnetic solutions. In Sec. IV these solutions are classified. These solutions as mentioned earlier yield generalizations of known magnetic solutions as well as a new class of magnetic solutions. After obtaining these solutions the electromagnetic fields seen by a locally nonrotating observer are also calculated.

II. THE MAGNETIZATION PRESCRIPTION

In this section, we summarize the magnetization procedure of Ernst applicable to any axially symmetric stationary solution of Einstein's equations. We do this in some detail to make the calculation self contained. The prescription involves making a Harrison-type transformation that generates in general new solutions of Einstein Maxwell equations from old ones. We confine our summary to cases where the starting solution is a vacuum metric. Following Ernst, any axially symmetric stationary solution may be written in the form

$$ds^2 = f^{-1}[-2P^{-2}\,d\xi d\xi^* + \rho^2 dT^2] - f(d\phi - \omega\, dT)^2, \tag{1}$$

where $f<0$. For any metric of the above type, the twist potential φ and the complex gravitational potential $\mathscr{E}$ are given by

$$\nabla\varphi = \frac{f^2}{\rho}(i\nabla\omega), \tag{2}$$

$$\mathscr{E} = f + i\varphi, \tag{3}$$

where[8]

$$\nabla \equiv \frac{\partial}{\partial\rho} + i\frac{\partial}{\partial z}. \tag{4}$$

To obtain the magnetized version of the metric given by Eq. (1) one replaces f and ω in Eq. (1) and f' and ω' defined by

$$f' = |\Lambda|^{-2} f, \tag{5}$$

$$\nabla\omega' = |\Lambda|^2\nabla\omega + \frac{\rho}{f}[\Lambda^*\nabla\Lambda - \Lambda\nabla\Lambda^*], \tag{6}$$

where[8]

$$\Lambda = 1 - \bar{B}_0^2\mathscr{E}. \tag{7}$$

The nontrivial part of the procedure is the integration of Eq. (6) to obtain ω'. For vacuum metrics—to which we confine

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ourselves—Eq. (6) is equivalent to

$$d(\omega' - \omega) = \overline{B}_0^4 [\varphi d\chi - (f^2 + \varphi^2) d\omega], \tag{8}$$

where χ is a new "potential" defined by

$$\nabla\chi = -\frac{i\rho}{f^2}\nabla(f^2 + \varphi^2). \tag{9}$$

It is simpler to obtain ω' from Eqs. (8) and (9) and we shall employ them in our computation. Under this transformation the complex gravitational potential $\mathscr{E}$ goes into $\mathscr{E}'$ where

$$\mathscr{E}' = \Lambda^{-1}\mathscr{E}, \tag{10}$$

whilst the complex electromagnetic potential changes from $\Phi = 0$ to Φ' given by

$$\Phi' = -\overline{B}_0\Lambda^{-1}\mathscr{E}. \tag{11}$$

It should be noted that in the axially symmetric stationary case $A_1 = A_2 = 0$. The complex electromagnetic field is related to the vector potential by

$$\Phi' = A_3 + iA'_0 \tag{12}$$

with A'_0 satisfying

$$\nabla A_0 + \omega'\nabla A_3 = i\rho f'^{-1}\nabla A'_0. \tag{13}$$

The combination Φ' is of interest since the electromagnetic field components as measured by a locally nonrotating observer is related to it by

$$B_{\hat\rho}^{NR} + iE_{\hat\rho}^{NR} = -P\frac{\partial\Phi'}{\partial z}, \tag{14a}$$

$$B_{\hat z}^{NR} + iE_{\hat z}^{NR} = P\frac{\partial\Phi'}{\partial\rho}. \tag{14b}$$

We now proceed to apply this method to the spacetimes of our interest.

III. APPLICATION TO STATIONARY CYLINDRICALLY SYMMETRIC METRICS

Following the treatment of Ref. (5) we see that all stationary cylindrically symmetric vacuum metrics may be taken in the form

$$ds^2 = e^{2\psi}(d\tau^2 + d\sigma^2) + \lambda_{00}\,dt^2 + 2\lambda_{01}\,dt\,d\phi + \lambda_{11}\,d\phi^2, \tag{15}$$

with

$$\lambda_\alpha = A_\alpha\tau^{1+b} + B_\alpha\tau^{1-b}, \tag{16}$$

$$e^{2\psi} = c\tau^{(b^2-1)}, \tag{17}$$

where α runs over (00), (01), (11) and $b^2 > 0$. The τ and σ are related to the usual coordinates ρ and z as

$$\tau = \sqrt{2}\rho, \quad \sigma = \sqrt{2}z, \tag{18}$$

while the A's and B's in Eq. (16) satisfy the normalization conditions

$$A_{00}A_{11} - A_{01}^2 = B_{00}B_{11} - B_{01}^2 = 0, \tag{19a}$$

$$A_{00}B_{11} + A_{11}B_{00} - 2A_{01}B_{01} = -\tfrac{1}{2}. \tag{19b}$$

The mass and angular momentum per unit value of z are given by

$$m = \tfrac{1}{4} + \tfrac{1}{2}b(A_{11}B_{00} - A_{00}B_{11}), \tag{20}$$

$$j = \tfrac{1}{2}b(A_{01}B_{11} - A_{11}B_{01}). \tag{21}$$

By comparing the metric given by Eq. (15), with the general form of Eq. (1) we identify

$$d\xi = \frac{d\rho + idz}{\sqrt{2}}, \tag{22}$$

$$P^{-2} = 2\lambda_{11}e^{2\psi}, \tag{23}$$

$$f = -\lambda_{11} = -(A_{11}\tau^{1+b} + B_{11}\tau^{1-b}), \tag{24}$$

$$\omega = -\frac{\lambda_{01}}{\lambda_{11}} = -\frac{A_{01}\tau^b + B_{01}\tau^{-b}}{A_{11}\tau^b + B_{11}\tau^{-b}}. \tag{25}$$

To proceed with the magnetization we first compute the twist potential φ using Eq. (2) which in component form reads

$$\frac{\partial\varphi}{\partial\rho} = -\frac{f^2}{\rho}\frac{\partial\omega}{\partial z}, \tag{26a}$$

$$\frac{\partial\varphi}{\partial z} = \frac{f^2}{\rho}\frac{\partial\omega}{\partial\rho}. \tag{26b}$$

Using

$$\frac{\partial\omega}{\partial z} = 0, \tag{27a}$$

$$\frac{\partial\omega}{\partial\tau} = \frac{2bD\tau}{\lambda_{11}^2}, \tag{27b}$$

where

$$D \equiv A_{11}B_{01} - A_{01}B_{11}, \tag{28}$$

it follows that

$$\varphi = 4bDz + c_1, \tag{29}$$

where c_1 is a (real) constant of integration. Consequently, from Eqs. (3), (5) and (7) we have

$$\mathscr{E} = -\lambda_{11} + i(4bDz + c_1), \tag{30}$$

$$f' = -\frac{\lambda_{11}}{|\Lambda|^2}, \tag{31}$$

where

$$\Lambda = (1 + \overline{B}_0^2\lambda_{11}) - i\overline{B}_0^2(4bDz + c_1) \tag{32}$$

We now proceed to the nontrivial part of the prescription viz., the determination of ω'. As mentioned earlier we do it via the potential χ. Equations (9) defining χ are the two partial differential equations

$$\frac{\partial\chi}{\partial\rho} = \frac{\rho}{f^2}\frac{\partial}{\partial z}(f^2 + \varphi^2), \tag{33a}$$

$$\frac{\partial\chi}{\partial z} = -\frac{\rho}{f^2}\frac{\partial}{\partial\rho}(f^2 + \varphi^2). \tag{33b}$$

In this case f and φ are given by Eqs. (24) and (29) so that Eqs. (33) become

$$\frac{\partial\chi}{\partial\tau} = \frac{4bD(4bDz + c_1)\tau^{2b-1}}{(A_{11}\tau^{2b} + B_{11})^2}, \tag{34a}$$

$$\frac{\partial\chi}{\partial z} = -2\left[1 + b - \frac{2bA_{11}}{(A_{11}\tau^{2b} + B_{11})}\right]. \tag{34b}$$

We shall now write Eq. (8) as

$$\frac{\partial}{\partial \rho}(\omega' - \omega) = \overline{B}_0^4\Big[\varphi \frac{\partial \chi}{\partial \rho} - (f^2 + \varphi^2)\frac{\partial \omega}{\partial \rho}\Big], \quad (35a)$$

$$\frac{\partial}{\partial z}(\omega' - \omega) = \overline{B}_0^4\Big[\varphi \frac{\partial \chi}{\partial z} - (f^2 + \varphi^2)\frac{\partial \omega}{\partial z}\Big]. \quad (35b)$$

Employing Eqs. (24), (27), (29) and (34), Eqs. (35) become after some simplification

$$\frac{\partial}{\partial \tau}(\omega' - \omega) = 2bD\overline{B}_0^4\Big[\frac{(4bDz + c_1)^2\tau^{2b-1}}{(A_{11}\tau^{2b} + B_{11})^2} - \tau\Big], \quad (36a)$$

$$\frac{\partial}{\partial z}(\omega' - \omega) = -2\overline{B}_0^4(4bDz + c_1) \times\Big(1 + b - \frac{2bB_{11}}{(A_{11}\tau^{2b} + B_{11})}\Big). \quad (36b)$$

Integrating Eq. (36b) gives

$$\omega' - \omega = -2\overline{B}_0^4 z(2bDz + c_1) \times\Big(1 + b - \frac{2bB_{11}}{(A_{11}\tau^{2b} + B_{11})}\Big) + c_1'(\tau). \quad (37)$$

Differentiating Eq. (37) w.r.t τ and using Eq. (36a) after some algebra and use of the normalization Eq. (19), we find $c_1'(\tau)$ satisfies the equation

$$\frac{\partial c_1'}{\partial \tau} = 2bD\overline{B}_0^4\Big[-\tau + \frac{c_1^2\tau^{2b-1}}{(A_{11}\tau^{2b} + B_{11})^2}\Big], \quad (38)$$

which can be integrated to yield

$$c_1' = -bD\overline{B}_0^4\Big[\tau^2 + 1\frac{c_1^2}{bA_{11}(A_{11}\tau^{2b} + B_{11})}\Big] + c_2, \quad A_{11} \neq 0, \quad (39)$$

where c_2 is another constant of integration. Equation (37) thus becomes

$$\omega' = \omega - \overline{B}_0^4\Big[bD\Big\{\tau^2 + 4z^2\Big(1 + b - \frac{2bB_{11}}{(A_{11}\tau^{2b} + B_{11})}\Big)\Big\} + 2zc_1\Big(1 + b - \frac{2bB_{11}}{(A_{11}\tau^{2b} + B_{11})}\Big) + \frac{Dc_1^2}{A_{11}(A_{11}\tau^{2b} + B_{11})}\Big] + c_2, \quad A_{11} \neq 0. \quad (40)$$

For completeness we mention that Eqs. (34) can be integrated to obtain the potential χ. The result is

$$\chi = -2z\Big(1 + b - \frac{2bA_{11}}{(A_{11}\tau^{2b} + B_{11})}\Big) - \frac{2Dc_1}{A_{11}(A_{11}\tau^{2b} + B_{11})} + c_2', \quad A_{11} \neq 0. \quad (41)$$

In the case $A_{11} = 0$ the equations are simpler and we only quote the results

$$\omega' = \omega - 2c_1\overline{B}_0^4(1 - b)z + c_2, \quad (42)$$

with

$$\chi = -2(1 - b)z + c_2'. \quad (43)$$

Having obtained f' and ω' the magnetization procedure is complete. The new magnetic solution as mentioned before is given by Eq. (1) with f and ω replaced by f' and ω' respectively and is of the form

$$ds^2 = |\Lambda|^2\Big[2e^{2\psi}(d\rho^2 + dz^2) - \frac{\rho^2}{\lambda_{11}}dt^2\Big] + \frac{\lambda_{11}}{|\Lambda|^2}(d\phi - \omega' dt)^2, \quad (44)$$

where

$$|\Lambda|^2 = (1 + \overline{B}_0^2\lambda_{11})^2 + \overline{B}_0^4(4bDz + c_1)^2. \quad (45)$$

IV. THE NEW MAGNETIZED SOLUTIONS

Before examining the new solutions, we make the following remarks which would clarify our classification of the solutions. From Eqs. (21) and (28) the angular momentum per unit length of the original metric is seen to be

$$j = -\tfrac{1}{2}bD. \quad (46)$$

Hence,

$$j = 0 \Leftrightarrow D = 0. \quad (47)$$

Further from Eqs. (28) and (19),

$$D^2 = -\tfrac{1}{4}A_{11}B_{11}. \quad (48)$$

Since $\lambda_{11} \neq 0$, it follows that $D = 0$ iff $A_{11} = 0$ or $B_{11} = 0$ but not both. It is easy to show with a little algebra that $D = 0$ iff $\omega =$ constant and consequently we have

$$j = 0 \Leftrightarrow D = 0 \Leftrightarrow \omega = \text{constant} \Leftrightarrow A_{11} = 0 \text{ or } B_{11} = 0 \text{ (but not both)}. \quad (49)$$

Further it is easy to see that any overall constant c_2 in ω' can be transformed away by a coordinate transformation of the type

$$\bar{\phi} = \phi - c_2 t, \quad \bar{t} = t. \quad (50)$$

Such a transformation does not affect the general form of the metric and hence there is no loss of generality in setting $c_2 = 0$ in Eqs. (40) and (42) for ω'. Also as expected if $\overline{B}_0 = 0$ in Eq. (45) one gets back to the original unmagnetized metric. With these remarks we classify the solutions as follows:

A. D = 0 solutions

As pointed out earlier these correspond to solutions with constant ω which can be transformed away by a suitable transformation of the form Eq. (50). Hence they correspond to magnetized versions of *static cylindrically symmetric solutions*. Moreover since $D = 0$, $\varphi = c_1$ and therefore in general these solutions have a constant twist. These solutions fall into two classes according to whether $B_{11} = 0$ or $A_{11} = 0$, each representative of two well known solutions as seen below.

1. Generalized Twisted Melvin Magnetic Universe (GTMMU)

This class corresponds to the particular choice of constants

$$B_{11} = 0, \quad A_{11} = \tfrac{1}{2}, \quad A_{01} = 0, \quad c = \tfrac{1}{2}, \quad (51)$$

which implies

$$B_{01} = 0 \quad \text{and} \quad B_{00} = -1. \quad (52)$$

For the original unmagnetized metric

$$m = \tfrac{1}{4}(1-b), \tag{53}$$

so that $m>0$ iff $b<1$. The metric is given by Eq. (45) where

$$\lambda_{11} = \frac{\rho^2}{\epsilon^2}, \quad e^{2\psi} = \tfrac{1}{2}\tau^{b^2-1}, \tag{54}$$

$$\omega' = -2\overline{B}{}_0^4 c_1(1+b)z, \tag{55}$$

$$|\Lambda|^2 = \left(1 + \frac{\overline{B}{}_0^2\rho^2}{\epsilon^2}\right)^2 + \overline{B}{}_0^4 c_1^2, \tag{56}$$

$$\epsilon^2 = \tau^{1-b}, \quad \epsilon > 0. \tag{57}$$

If $b=1$ and $c_1 = 0$ we get the well known Melvin Magnetic Universe (MMU) whereas if $b=1$ and $c_1 \neq 0$ we get the Twisted Melvin Magnetic Universe (TMMU) discussed by Harrison.[1] In the case of a "twisted" solution the angular velocity ω' varies montonically with z becoming 0 on the $z=0$ plane. The case $b\neq 1, c_1\neq 0$ corresponds to a Generalized Melvin Magnetic Universe (GMMU).

It should be noticed that other solutions, e.g., $B_{11} = B_{01} = 0, A_{11} = \tfrac{1}{2}, B_{00} = -1, c = \tfrac{1}{2}$, but $A_{01}\neq 0$, $A_{00}\neq 0$ can be transformed into a suitable one of the above solutions by a coordinate transformation of type Eq. (50). Similarly, solutions with $A_{11}\neq \tfrac{1}{2}, B_{11}\neq -1$, can be transformed by a scaling of coordinates *t and* ϕ.[9]

2. Twisted Weyl–Levi-Cività magnetic solution (TWLCM)

It is known that the special case

$$A_{11} = 0, \quad B_{11} = \tfrac{1}{2}, \quad B_{01} = 0, \tag{58}$$

corresponds to the Weyl–Levi-Cività solution whose source is an infinite rod of linear density

$$m = \tfrac{1}{4}(1+b), \tag{59}$$

or the infinite mass cylindrical limit of the augmented Schwarzschild solution.[10] Our solution for the above choice of parameters has a constant twist c_1 leading to the above nomenclature. In this case

$$\lambda_{11} = \frac{\rho^2}{\bar\epsilon^2}, \quad e^{2\psi} = c\tau^{b^2-1}, \tag{60}$$

$$\omega' = -2\overline{B}{}_0^4 c_1(1+b)z, \tag{61}$$

$$|\Lambda|^2 = \left(1 + \frac{\overline{B}{}_0^2\rho^2}{\bar\epsilon^2}\right)^2 + \overline{B}{}_0^4 c_1^2, \tag{62}$$

where

$$\bar\epsilon^2 = \tau^{1+b}, \quad \bar\epsilon > 0. \tag{63}$$

If $c_1 = 0$ we get the Weyl–Levi-Cività magnetic solution (WLCM). The particular case $b=1$ corresponds to the cylindrical limit of the Schwarzschild solution and in this case

$$\lambda_{11} = \tfrac{1}{2}, \quad e^{2\psi} = c \tag{60a}$$

$$\omega' = -4\overline{B}{}_0^4 c_1 z, \tag{61a}$$

$$|\Lambda|^2 = [1 + (\overline{B}{}_0^2/2)]^2 + \overline{B}{}_0^4 c_1^2, \tag{62a}$$

with

$$\bar\epsilon = \tau. \tag{63a}$$

In this case if $c_1 = 0$ then the new solution is reducible to the original solution by a scaling of coordinates so that the Harrison transformation is an identity transformation. The peculiarity of this case will be more apparent when we look at the electromagnetic fields.

As before if $B_{01}\neq 0$ the solution may be transformed to one of the above by a coordinate transformation. We thus find that under the magnetization procedure the static solutions retain their original cylindrical symmetry only if there is no twist, i.e., $c_1 = 0$.

B. $D\neq 0$, $c_1 = 0$ solution

In this case the original background is stationary. ω is not a constant and therefore cannot be globally transformed away. These solutions correspond to nonvanishing j and consequently the magnetized version of these solutions form a new class of solutions. The relevant function for this metric are given by

$$\lambda_{11} = A_{11}\tau^{1+b} + B_{11}\tau^{1-b}, \tag{64}$$

$$\omega' = \omega - bD\overline{B}{}_0^4\left[\tau^2 + 4z^2\left(1 + b - \frac{2bB_{11}}{(A_{11}\tau^{2b} + B_{11})}\right)\right], \tag{65}$$

$$|\Lambda|^2 = (1 + \overline{B}{}_0^2\lambda_{11})^2 + 16b^2D^2\overline{B}{}_0^4 z^2. \tag{66}$$

We thus see that for this class of solutions even though $c_1 = 0$ the metrics are axially symmetric and have lost their original cylindrical symmetry under the magnetization transformation. Moreover, these solutions have reflection symmetry with respect to $z=0$ plane (the z^2 dependence) and are qualitatively very different from the previously discussed metrics with constant twist.

C. $D\neq 0$, $c_1\neq 0$ solution

These correspond to the most general of our solutions and represent "twisted" versions of the previous solutions. The most general form is given by Eq. (45) with ω' given by Eq. (40).

V. THE ELECTROMAGNETIC FIELDS

We conclude this investigation by computing the electromagnetic fields generated by the Harrison type transformation or in other words the magnetization prescription. As mentioned in Sec. II the fields as seen by a locally nonrotating observer (LNRO) are directly calculable from the potential Φ'. For any metric of the form given by Eq. (1) a LNRO is specified by a tetrad whose nonvanishing components are given by

$$e_{\hat 0}{}^{t} = f^{1/2}\rho^{-1}, \quad e_{\hat 1}{}^{\rho} = e_{\hat 2}{}^{z} = f^{1/2}P, \quad e_{\hat 3}{}^{\phi} = f^{-1/2}, \quad e_{\hat 0}{}^{\phi} = \omega e_{\hat 0}{}^{t}. \tag{67}$$

Employing Eqs. (3) and (7) in Eq. (11), Eq. (12) yields in general

$$A_3 = -\frac{\overline{B}_0[f - \overline{B}{}_0^2(f^2 + \varphi^2)]}{|\Lambda|^2}, \tag{68a}$$

$$A_0' = -\frac{\overline{B}_0\varphi}{|\Lambda|^2}, \tag{68b}$$

which for our specific case gives

$$A_3 = \frac{\overline{B}_0[\lambda_{11} + \overline{B}{}_0^2(\lambda_{11}^2 + \varphi^2)]}{|A|^2}, \tag{69a}$$

$$A_0' = -\frac{\overline{B}_0\varphi}{|A|^2}, \tag{69b}$$

with φ and $|A|^2$ being given by Eqs. (29) and (45) respectively.

Using Eqs. (14), (12) and (69), after a straightforward computation the electromagnetic field components may be written down in all the cases as

$$B_{\hat\rho}^{\rm NR} = -\frac{8bDP\varphi(1 + \overline{B}{}_0^2\lambda_{11})}{|A|^4}, \tag{70a}$$

$$B_{\hat z}^{\rm NR} = \frac{\sqrt{2}\overline{B}_0P\lambda_{11},_\tau[(1 + \overline{B}{}_0^2\lambda_{11})^2 - \overline{B}{}_0^4\varphi^2]}{|A|^4}, \tag{70b}$$

$$E_{\hat\rho}^{\rm NR} = \frac{4bD\overline{B}_0P[(1 + \overline{B}{}_0^2\lambda_{11})^2 - \overline{B}{}_0^4\varphi^2]}{|A|^4}, \tag{70c}$$

$$E_{\hat z}^{\rm NR} = 2\frac{\sqrt{2}\overline{B}{}_0^3P\lambda_{11,\tau}\varphi[1 + \overline{B}{}_0^2\lambda_{11}]}{|A|^4}, \tag{70d}$$

with λ_{11}, P, D, φ and $|A|^2$ as defined by Eqs. (16), (23), (28), (29) and (45) respectively. To see the structure more clearly we write down the particular forms of the fields in each of the cases discussed in the previous section.

A1. GTMMU fields

In this case,

$$B_{\hat\rho}^{\rm NR} = E_{\hat\rho}^{\rm NR} = 0,$$

and

$$B_{\hat z}^{\rm NR} = (1 + b)\overline{B}_0\epsilon^b\frac{[(1 + \overline{B}{}_0^2\rho^2/\epsilon^2)^2 - \overline{B}{}_0^4c_1^2]}{[(1 + \overline{B}{}_0^2\rho^2/\epsilon^2)^2 + \overline{B}{}_0^4c_1^2]^2}, \tag{71a}$$

$$E_{\hat z}^{\rm NR} = \frac{2(1 + b)\overline{B}{}_0^3c_1\epsilon^b(1 + \overline{B}{}_0^2\rho^2/\epsilon^2)}{[(1 + \overline{B}{}_0^2\rho^2/\epsilon^2)^2 + \overline{B}{}_0^4c_1^2]^2}. \tag{71b}$$

Although the metric in general has a z dependence the electromagnetic fields in a LNRF are independent of z. Further as $\rho \to \infty$, $B_{\hat z}^{\rm NR} \sim \tau^{-(b^2+3b+4)}$, and $E_{\hat z}^{\rm NR} \sim \tau^{-(b^2+5b+6)}$ so that as $\rho \to \infty$, $B_{\hat z}^{\rm NR}$ and $E_{\hat z}^{\rm NR} \to 0$. It is interesting to note that as $\rho \to 0$, $B_{\hat z}^{\rm NR} \sim \tau^{b(1-b)/2}$, $E_{\hat z}^{\rm NR} \sim \tau^{b(1-b)/2}$ so that for solutions with linear density positive, $m > 0$, i.e., $b < 1$, both $B_{\hat z}$ and $E_{\hat z}$ vanish on the axis.[11]

A2. TWLCM fields

In this case too

$$B_{\hat\rho}^{\rm NR} = E_{\hat\rho}^{\rm NR} = 0, \tag{72a}$$

while

$$B_{\hat z}^{\rm NR} = \frac{(1 - b)\overline{B}_0}{\overline{\epsilon}^b}\frac{[(1 + \overline{B}{}_0^2\rho^2/\epsilon^2)^2 - \overline{B}{}_0^4c_1^2]}{[(1 + \overline{B}{}_0^2\rho^2/\overline{\epsilon}^2)^2 + \overline{B}{}_0^4c_1^2]^2}, \tag{72b}$$

$$E_{\hat z}^{\rm NR} = \frac{2(1 - b)\overline{B}{}_0^3c_1}{\overline{\epsilon}^b}\frac{(1 + \overline{B}{}_0^2\rho^2/\overline{\epsilon}^2)}{[(1 + \overline{B}{}_0^2\rho^2/\overline{\epsilon}^2)^2 + \overline{B}{}_0^4c_1^2]^2}. \tag{72c}$$

Here also the fields are z independent and as $\rho \to \infty$, $B_{\hat z}^{\rm NR} \sim \tau^{-[(b-3/2)^2 + 7/4]} \to 0$. However $E_{\hat z}^{\rm NR} \sim \tau^{-(b-3)(b-2)}$ so that $E_{\hat z}^{\rm NR} \to 0$ unless $2 < b < 3$. In this case on the axis the fields are divergent. $b = 1$ is a degenerate case as seen from, Eq. (72), since there are no electromagnetic fields. Thus the cylindrical limit of the Schwarzschild solution under the Harrison transformation does not pick up any electromagnetic field but a constant vector potential even if $c_1 \neq 0$.

In both the cases it should be noted that if the constant twist $c_1 = 0$ then only a magnetic field in the z direction is produced but no electric field. The magnetic field for GMMU is given by

$$B_{\hat z}^{\rm NR} = \frac{(1 + b)\overline{B}_0\epsilon^b}{(1 + \overline{B}{}_0^2\rho^2/\epsilon^2)^2}, \tag{73}$$

which for MMU becomes

$$B_{\hat z}^{\rm NR} = \frac{2\overline{B}_0}{(1 + \overline{B}{}_0^2\rho^2)^2}, \tag{74}$$

whilst that of WLCM becomes

$$B_{\hat z}^{\rm NR} = \frac{(1 - b)\overline{B}_0}{\overline{\epsilon}^b(1 + \overline{B}{}_0^2\rho^2/\overline{\epsilon}^2)^2}. \tag{75}$$

For the TMMU however

$$B_{\hat z}^{\rm NR} = 2\overline{B}_0\frac{[(1 + \overline{B}{}_0^2\rho^2)^2 - \overline{B}{}_0^4c_1^2]}{[(1 + \overline{B}{}_0^2\rho^2)^2 + \overline{B}{}_0^4c_1^2]^2}, \tag{76a}$$

$$E_{\hat z}^{\rm NR} = \frac{4\overline{B}{}_0^3c_1(1 + \overline{B}{}_0^2\rho^2)^2}{[(1 + \overline{B}{}_0^2\rho^2)^2 + \overline{B}{}_0^4c_1^2]^2}. \tag{76b}$$

B. $D \neq 0$, $c_1 = 0$

The structure of the electromagnetic fields in this case is very different from the previous case because of the nontrivial rotation present in the metric. The fields are no longer z independent and are given as

$$B_{\hat\rho}^{\rm NR} = -\frac{32b^2D^2\overline{B}{}_0^3P(1 + \overline{B}{}_0^2\lambda_{11})z}{|A|^4}, \tag{77a}$$

$$B_{\hat z}^{\rm NR} = \frac{\sqrt{2}\overline{B}_0P\lambda_{11,\tau}[(1 + \overline{B}{}_0^2\lambda_{11})^2 - 16b^2D^2\overline{B}{}_0^4z^2]}{|A|^4}, \tag{77b}$$

$$E_{\hat\rho}^{\rm NR} = \frac{4bD\overline{B}_0P[(1 + \overline{B}{}_0^2\lambda_{11})^2 - 16b^2D^2\overline{B}{}_0^4z^2]}{|A|^4}, \tag{77c}$$

$$E_{\hat z}^{\rm NR} = \frac{8\sqrt{2}bD\overline{B}{}_0^3P\lambda_{11,\tau}(1 + \overline{B}{}_0^2\lambda_{11})z}{|A|^4}. \tag{77d}$$

It is easy to see that in this case also the fields vanish as $\rho \to \infty$ for a fixed value of z. Using the tetrad components listed in Eq. (67) and its inverse the coordinate components of the electromagnetic field tensor may be obtained. We have done this and verified explicitly that these fields satisfy the generally covariant Maxwell's equations without any sources.

However, no one of the above solutions correspond to the asymptotic form of the magnetized Kerr solution.

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