Observable Dirac-Type Singularities in Berry's Phase and the Monopole

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A three-dimensional generalization of the sign-change (π phase shift) rule for adiabatic cycles of spin-$1/2$ or two-state wave functions encircling a degeneracy in the parameter space of the Hamiltonian yields a Dirac-type singularity wherein any closed circuit of the adiabatic cycle in which the degeneracy is "looped" results in an observable $\pm 2\pi$ phase shift. It is concluded that an interferometer loop similarly taken around a magnetic monopole of strength $n/2$ yields an observable $\pm 2n\pi$ phase shift, n being an integer.

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1. Introduction.— In 1959, Aharonov and Bohm [1] made the important observation that a topological phase factor $e^{i\phi}$ picked up by the wave function of an electron in a closed circuit around a magnetic field, introduced by Dirac in his 1931 paper on the monopole [2], is a measurable physical effect, ϕ being proportional to the magnetic flux through the circuit. Wu and Yang [3] conjectured that only the phase factor $e^{i\phi}$ and not the phase ϕ itself is measurable, implying that ϕ is defined only modulo 2π. More recently, in another well-known work, Berry [4] discovered a topological phase (*geometric phase*) in the evolution of a quantum system under the action of a cyclic, adiabatic Hamiltonian and related it to the Aharonov-Bohm phase. Berry's phase, as well as the nonadiabatic geometric phases discovered by Pancharatnam [5], Aharonov and Anandan [6], Samuel and Bhandari [7], etc., have implicitly or explicitly been defined as modulo 2π quantities.

A different perspective on the geometric phase has been brought out in a series of experimental and theoretical contributions by the present author [8–14]. This work uses interference of polarized light, exploits the mathematical isomorphism between polarization of light and the two-state quantum system, and shows through experiments that (i) a continuously measured geometric phase shift is unbounded, as opposed to modulo 2π, and can be nonintegrable on the parameter space of the experiment [8], (ii) a geometric phase shift as defined by the Pancharatnam criterion [5] can have discontinuous jumps and can change sign for small variation in the parameters near singular points (or lines, surfaces) in the parameter space where the two interfering states become orthogonal [9,11,12], and (iii) a circuit in parameter space enclosing several such singularities results in a measurable phase shift $\int d\phi$ equal to 2π times the algebraic sum of the strengths of the singularities [10,12], hence the term "Dirac singularities" [2]. These results have a bearing on the question of observability of $2n\pi$ phase shifts and add a new dimension to the 4π spinor symmetry problem [11].

The above results were obtained in the context of nonadiabatic quantum evolution [6], using Pancharatnam's definition of phase difference between different states [5]. In a geometric description, the phase jumps are easily understood in terms of the "shortest geodesic rule" for closing open paths in the state space [9,15]. In this paper, with the help of a model problem very close to the original adiabatic setting in which Berry's geometric phase was arrived at [4,16], we show that an observable Dirac-type singularity is also inherent in the adiabatic geometric phase which is like the phase acquired by a charged particle going around a loop in the field of a magnetic monopole [4].

2. The Model Problem.— A beam of quantum mechanical spin-$1/2$ particles (electrons or neutrons), in a spin state $|\psi_i\rangle$, enters a ring-shaped configuration of paths at some point i (Fig. 1) such that it has a choice of two paths 1 or 2 through one or the other identical halves of the ring, under the action of spin-Hamiltonians H_1 or H_2 and exits at the

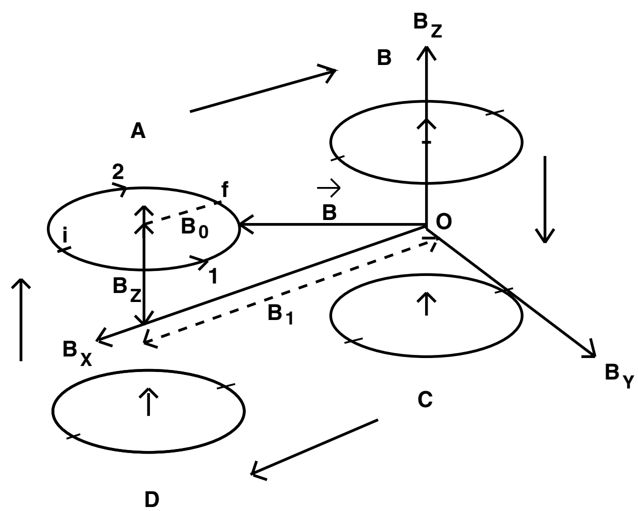

FIG. 1. For a given set of parameters b_1 and b_z, the propagating particle sees a spin Hamiltonian corresponding to a magnetic field rotating in one sense, represented by the semicircle $i1f$, if it goes through the right half of the ring and to one rotating in the opposite sense, i.e., $i2f$ if it goes through the left half. A change in B_1 or B_z, or both, moves the circle $i1f2i$, resulting in general in a change in the final states $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$, in the phase difference α between them and consequently in the current J.

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diametrically opposite point f in state $|\psi_{f1}\rangle$ or $|\psi_{f2}\rangle$. The latter, for $m = 1, 2$, are given by

$$|\psi_{fm}\rangle = U_m|\psi_i\rangle = T\exp\left[(-i/\hbar)\int H_m(t)\,dt\right]|\psi_i\rangle . \tag{1}$$

Here $H_m(t) = \mu[\vec{\sigma}\cdot\vec{B}_m(t)]$, μ is the gyromagnetic ratio and $\vec{\sigma}$ is the vector of Pauli matrices. The magnetic field $\vec{B}_m(t)$ seen by the particle at time t consists of two parts: (1) a constant field with components B_1 and B_z in the x and z directions, respectively, and (2) a rotating field in the x-y plane with constant magnitude B_0 and frequency ω, ω being such that $\omega T = \pm\pi$; T being the traversal time for paths 1 and 2. The rotation has the opposite sense in paths 1 and 2. Thus in Eq. (1), $B_{mx} = (B_1 + B_0\cos\omega t)$, $B_{mz} = B_z$, $B_{1y} = (B_0\sin\omega t)$, and $B_{2y} = (-B_0\sin\omega t)$. For $B_z = 0$, the problem reduces to that studied by Geller [17] who proposed a mesoscopic physics experiment to detect an abrupt π phase jump resulting from an adiabatic circuit encircling a degeneracy in the parameter space of the Hamiltonian; the "rotating magnetic field" term in their problem arising from spin-orbit interaction of the electrons. The current J at the other end of the ring is proportional to $|\langle\psi|\psi\rangle|^2$, where $|\psi\rangle = (|\psi_{f1}\rangle + |\psi_{f2}\rangle)$. The term of interest in the expression for J is the modulus c and phase α of the interference term $2\langle\psi_{f2}|\psi_{f1}\rangle$, the Pancharatnam phase difference between $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$. In general, both c and α vary as the parameters in H_1 and H_2 are varied. Experimentally, α can be determined by introducing a variable, state-independent phase difference between the two halves of the ring until J is maximum. This is routinely done in neutron interferometry by rotating a phase shifter in the path of one or both the beams, e.g., in recent geometric phase experiments [18].

Let us define dimensionless variables $b_1 = B_1/B_0$, $b_z = B_z/B_0$, $\beta = \mu B_0/\hbar\omega$ and $\gamma = \mu B_z/\hbar\omega$. Consider an experiment in which the variables b_1 and b_z are varied along a path such as $ABCDA$, $EFGHE$, or $SPQRS$ in Fig. 2 by appropriate variation of the fields B_1 and B_z, while the phase difference α is being continuously monitored. When $B_z = 0$ and $B_1 = \pm B_0$, i.e., the points S_1 and S_2 in Fig. 2, the adiabatic cycle passes through the point of degeneracy (the point O in Fig. 1) and these are the singular points. Crossing of any one of these points by the adiabatic cycle results in a phase jump of magnitude π. However, in this case there is inevitable departure from adiabatic evolution near the singularities. Exactly at the singularity, the two final states $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$ are orthogonal. This is the case studied in Ref. [17].

When $B_z \neq 0$, a closed nonadiabatic solution for the evolution of the wave functions along paths 1 and 2 under the above Hamiltonian does not exist. However, for the central result of this paper, we do not need the nonadiabatic solution. When the circuit in the parameter space stays a finite distance away from the singularity, it is possible to

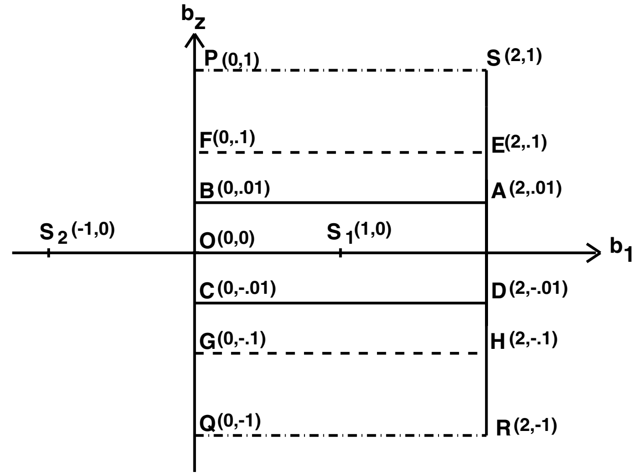

FIG. 2. Circuits in the parameter space b_1, b_z (not to scale) chosen for computing phase shifts shown in Fig. 3. S_1 and S_2 are singular points where $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$ become orthogonal and the phase between the two beams becomes undefined. For adiabatic evolution, a counterclockwise (clockwise) circuit enclosing the singularity S_1 gives a phase change -2π $(+2\pi)$.

choose ω small enough so that the evolution is adiabatic over the entire cycle all along the circuit. For simple circuits of the kind considered (Fig. 2), this is true if $\gamma \gg 1$. In the limit $\gamma \to \infty$, the two final states $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$ are the same and the dynamical phase in the two paths exactly cancels, making the problem equivalent to that of a single full cycle along the ring with the dynamical phase subtracted [4]. The phase α, a purely geometric phase, is then given by the solid angle subtended by the adiabatic cycle at the degeneracy. If one considers the evolution of the projection of the cycle on a unit sphere centered at the degeneracy, it is easy to see that any closed circuit of the adiabatic cycle of the type shown in Fig. 2, i.e., a circuit that "loops" the degeneracy, sweeps an area equal to the entire sphere, i.e., a solid angle $\pm 4\pi$, the sign depending upon the sense of traversal of the circuit. For circuits such as $ABCDA$ and $EFGHE$ which pass close to the singularity, the sharp variation of the area of the projected cycle on the unit sphere when the periphery of the adiabatic cycle is closest to the degeneracy (the point O in Fig. 1) can be intuitively seen. The change in the sense of this variation with the sign of B_z can also be visualized easily. The resulting phase shifts are then defined unambiguously and do not have a "modulo 2π" ambiguity at any stage.

We have carried out numerical simulations of the actual evolution of the wave functions along paths 1 and 2 under the action of the Hamiltonian H_m and computed the variation of α, i.e., the quantity $\int d\alpha$, along a few circuits in the space of parameters b_1 and b_z, e.g., the circuits $ABCDA$, $EFGHE$, and $SPQRS$ in Fig. 2. Along each of the four segments of a rectangular circuit, 100 equally spaced points are chosen and for each value of b_1 and b_z, the unitary time evolution operator for a time interval $\delta t = \pi/20\,000\omega$ is computed at each of 20 000 equispaced points along paths 1 and 2. The adiabaticity parameter β has been chosen so that $\gamma = (b_z \beta) = 20$ for all the circuits, i.e., for the circuit $ABCDA$, $\beta = 2000$, for $EFGHE$, $\beta = 200$ and for $SPQRS$, $\beta = 20$. This choice ensures that the relevant adiabaticity parameter is larger than 20 all along the cycle everywhere on each circuit. The products of the 20 000 U matrices along each path are then computed to yield U_m, which, multiplied with the initial wave function $|\psi_i\rangle$ [taken to be an eigenstate of $H_m(0)$], yields the final states $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$. The phase difference α is then computed according to the expressions given above. In the adiabatic limit the dynamical phase in the two paths is exactly compensated and α reflects the geometric phase difference.

Figure 3 shows the results of the computation. The computed variation of α shows all the expected features mentioned above. The net phase change equal to -2π, the sharp phase jumps with the expected relative sign at the two points where the cycle passes close to the degeneracy (circuits $ABCDA$ and $EFGHE$), the increase in sharpness of the jumps for a closer approach to the degeneracy and the smooth variation of the phase when the cycle always remains far from the degeneracy ($SPQRS$) are all clearly seen. We have verified that the sign of the phase shifts reverse with the sense of traversal of the circuits. For several circuits that do not enclose the singularity, the computed phase change is found to be zero. It is also found that as γ increases, the final states $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$ are closer to each other and that all the three curves shown in Fig. 3 approach those for the variation of the solid angle subtended by the cycle at the degeneracy.

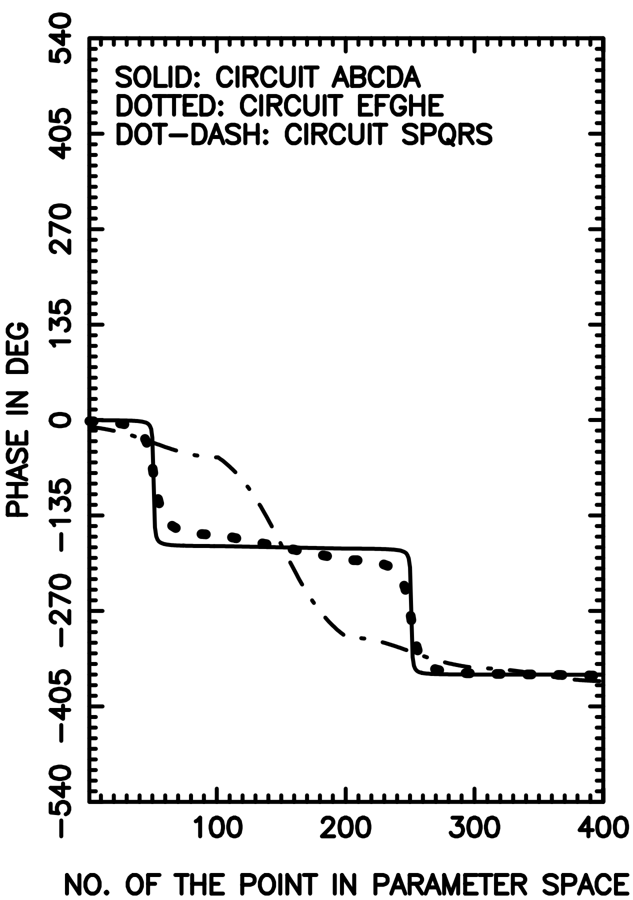

FIG. 3. The solid, the dotted, and the dot-dashed lines show the change in phase difference α between the two beams in final states $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$, as the parameters b_1 and b_z are varied along the circuits $ABCDA$, $EFGHE$, and $SPQRS$ (Fig. 2), respectively. Each segment of the circuit is divided into 100 equispaced points for computation. Note the total phase shift equal to -2π and the sharp variation of phase when the circuit passes close to the singularity, i.e., near the points 50 and 250 for circuits $ABCDA$ and $EFGHE$.

The above results suggest a useful three-dimensional generalization of the sign-change rule. If the adiabatic cycle in the parameter space is taken through a closed circuit such that it encircles the degeneracy in the process, the wave function acquires a $+2\pi$ or -2π phase change depending upon the sense of the circuit. If it does not encircle the degeneracy, the phase change is zero. The generalization suggested by Berry [4], based on Stone's result [19], refers to a segment such as PQ (with $P \to \infty, Q \to -\infty$) in Fig. 2. The present one is a more complete statement and may find uses in molecular problems. We also note that for the special case of the present problem ($B_z = 0$) studied by Geller [17] our results agree with theirs. We have shown that for $B_z \neq 0$, for the same direction of motion of the loop, their π phase jump has a negative sign for $B_z > 0$ and a positive sign for $B_z < 0$. This is nontrivial. One could, for example, set up a contraption such that a $+\pi$ phase shift accumulated in an electronic register activates a switch that triggers an explosive that kills a cat, while a $-\pi$ phase shift does not.

3. The Monopole.—Motivated by the above results, consider the following gedanken experiment with a Dirac monopole. Take a current loop in the x-y plane, divided into two halves similar to those in Fig. 1, which now represents the real space with space coordinates x, y, z replacing the magnetic field components B_x, B_y, B_z. Let the radius of the loop be B_0, the x coordinate of its center be B_1, and its height above the x-y plane be B_z. A current of charged spinless quantum particles enters at point i, has a choice of two paths 1 and 2, and is recombined at the point f where the phase difference α between the two complex transmission amplitudes a_1 and a_2 is measured by an interference experiment. Let a magnetic monopole of strength $-1/2$ be located at the point O (Fig. 1) and the current loop be transported along the path $ABCDA$ (or $EFGHE$ or $SPQRS$) as shown, while α is being continuously monitored and the orientation of the loop in space kept unaltered. It is easy to convince oneself that the variation in α due to the changing magnetic flux through the loop (Aharonov-Bohm effect), given by half the solid angle subtended by the loop at O, would be similar to that shown in Fig. 3, implying Dirac-type singularities in the b_1-b_z plane at the points S_1 and S_2 (Fig. 2). If, as usually assumed, the magnetic monopole has a string attached to it, there would be an additional $+2\pi$ phase jump when the loop encircles the string. This would be infinitely sharp hence unobservable if the string is infinitely thin but would be observable, leading to a net zero phase change for the circuit, if the string had a finite thickness.

The above results have been obtained using spin-$1/2$ wave functions evolving under an $\vec{S} \cdot \vec{B}$ Hamiltonian, $\vec{S}$ being the spin vector. However, it is an exact mathematical result [4] that the phase change of a wave function with spin component $n/2$ along the field direction, evolving under the same Hamiltonian, is exactly n times that for a wave function with spin component $1/2$. We conclude therefore that the phase change measured by the loop taken around a monopole of strength $n/2$, with an infinitely thin string, equals $\pm 2n\pi$, which is thus intrinsic to the problem and cannot be truncated to its modulo 2π value which is zero.

It may also be useful to note that for particles with spin quantum number $n/2$, with $n > 1$ and for problems involving more than two quantum states, the monopole picture of the geometric phase would not be valid for arbitrary Hamiltonians. One could, however, expect measurable phase jumps equal to $\pm n\pi$ in general.

4. The proposed neutron experiment.—A set of two counterrotating magnetic fields in the two arms of a neutron interferometer in planes normal to the beams can be set up with the technique used in the polarimetric experiment of Bitter and Dubbers [20], along with a uniform but variable magnetic field B_1 normal to the plane of the interferometer and a magnetic field B_z along each of the beams. The phase shifts as a function of B_1 and B_z can be measured with the technique used in Ref. [18]. The measurement of the current J, i.e., the flux of neutrons in the recombined beam as a function of B_1 and B_z, is of course straightforward. It is also important to note that the phase difference between $|\psi_{f1}\rangle$ and $|\psi_{f2}\rangle$ at each point in the parameter space is determined modulo 2π. For large β, therefore, it is very sensitive to small fractional errors in the dynamical phase in path 1 or 2. The basic topological effect can, however, be seen for smaller values of β by the choice of a circuit that does not pass too close to the singularity.

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