Density-matrix approach to a strongly coupled two-component Bose-Einstein condensate

Andal Narayanan*
Non-Accelerator Particle Physics Group, Indian Institute of Astrophysics, Koramangala, Bangalore 560 034, India

Hema Ramachandran
Raman Research Institute, Sadashivnagar, Bangalore 560 080, India

(Received 29 November 1999; published 12 October 2000)

The time evolution equations for average values of population and relative phase of a strongly coupled two-component Bose-Einstein condensate (BEC) are derived analytically. The two components are two hyperfine states, which are coupled by an external laser that drives fast Rabi oscillations between these states. Specifically, this derivation incorporates the two-mode model proposed in J. Williams *et al.*, e-print cond-mat 9904399 for the strongly coupled hyperfine states $|1,-1\rangle$ and $|2,1\rangle$ of ^{87}Rb. The fast Rabi cycles are averaged out and the rate equations so derived represent the slow dynamics of the system. These include the collapse and revival of Rabi oscillations and their dependence on detuning and trap displacement as reported in experiments of J. Williams. A procedure for stabilizing vortices is also suggested.

PACS number(s): 03.75.Fi, 74.50.+r, 32.80.Pj, 05.30.Jp

I. INTRODUCTION

Observations of Bose-Einstein condensation in trapped dilute alkali-metal atoms have opened up both experimental and theoretical challenges to understand the properties of such systems. The dynamical properties of a single condensate such as its collective excitations in a trap due to a time dependent drive has been one such area of research [1,2]. The experimental realization of simultaneous creation and confinement of Bose-Einstein condensates (BECs) in several hyperfine states of a given species of atom [3–5] has led to investigations on dynamics of two or more overlapping condensates by coupling them with an externally applied laser field. In particular experimental realization of binary mixtures of two hyperfine states namely, $|1,-1\rangle$ and $|2,1\rangle$ of ^{87}Rb has established the following properties of this coupled condensed system [6,7]:

(i) These two states have magnetic moments which are same to the first order. However, due to other small effects such as gravity, the nuclear magnetic moment and nonlinearity in the Zeeman shifts, the location of minima for the two states in the trap can be adjusted to be slightly different or exactly coincident.

(ii) Spontaneous interconversion from one state to the other is not seen due to the large difference in internal energies between these two states. The hyperfine energy is 6.8 GHz. This makes the two condensates distinguishable. These can be selectively imaged by choice of an appropriate laser.

(iii) These condensate states possess a relative quantum phase that can be measured. This phase evolves with time the rate being proportional to the chemical potential difference between the two condensates.

(iv) An external laser drive couples these two systems and helps to coherently transfer population from one state to the other.

The mixed condensates thus offer an ideal experimental apparatus to look for macroscopic realizations of dynamical effects like standard Josephson effects [8].

Theoretical calculations on such Josephson like oscillations in these coupled boson Josephson junctions (BJJ) [9,10] have shown several interesting dynamical effects. Recently, however [11], an experimental observation of an unexpected behavior of these coupled systems was reported. In the limit of sustained and large field strengths of the external coupling laser, that is when Ω, the Rabi frequency, was five to ten times larger than the trap frequency in the vertical direction, along which the two condensates sit displaced [12], the Rabi oscillations between the hyperfine states was found to collapse and revive. This occurred on a time scale which is large compared to the Rabi period. These slow varying modulations of the fast Rabi oscillations vanish at zero trap displacement. These were also seen to vanish when $\delta=0$, where $\delta=\omega-\omega_d$ is the detuning of the external laser frequency (ω_d) from the transition frequency (ω) between the hyperfine states. It was shown subsequently in the same paper that this phenomenon was due to a weak coupling between the low lying motional states of the trap. In particular a simplified two-mode model was suggested. In this two-mode model the trap ground state and first excited dipole state were coupled and couplings to all higher motional states were neglected. Thus Ref. [11] demonstrates the possibility of quantum state engineering of topological excitations, through the interplay between the internal and motional degrees of freedom of a BEC in a time orbiting potential trap. Numerical simulations by solving the Gross-Pitaeskii (GP) equations for the coupled system were carried out in [11], which reproduced the experimental features.

In the present paper, we derive the essential experimental features analytically, using the density-matrix approach. Equations for the fractional population (Z) in the hyperfine states and their relative phase (θ) as a function of time have been obtained. Averaging over the Rabi period, these equations represent the slow dynamics of the system. Collapse and revivals of Rabi oscillations and their dependence on detuning and trap displacement are seen to match qualita-

*Email address: andal@rri.ernet.in

tively with the experimental results described in [11]. A proposal for combining this strongly coupled regime to the weakly coupled Josephson regime is presented and its role in increasing the stability of vortices [14–16] is also speculated.

II. A DENSITY-MATRIX METHOD

The total wave-function of the two-component condensate is denoted by $\psi(r,t)$. Initially this wave-function just represents the total population (N_T) in the ground state $|1,-1\rangle$. An external laser drives the transition from this state to the $|2,1\rangle$ state coherently. So we can write

$$\psi(r,t)=\psi_1(r,t)+\psi_2(r,t).$$

These two states are of the form [11]

$$|\psi_1\rangle=(\alpha_1(t)c_0(t)|\phi_0\rangle+\alpha_2(t)d_1(t)|\phi_1\rangle)|1\rangle, \tag{1}$$

$$|\psi_2\rangle=(\alpha_2(t)c_0(t)|\phi_0\rangle+\alpha_1^*(t)d_1(t)|\phi_1\rangle)|2\rangle. \tag{2}$$

Here $|1\rangle$ and $|2\rangle$ refer to the hyperfine (internal) states and $|\phi_0\rangle$ and $|\phi_1\rangle$ refer to the motional (external) states, and

$$\alpha_1(t)=\cos\left(\frac{\Omega_{eff}t}{2}\right)-i\left(\frac{\delta}{\Omega_{eff}}\right)\sin\left(\frac{\Omega_{eff}t}{2}\right),$$

$$\alpha_2(t)=-i\left(\frac{\Omega}{\Omega_{eff}}\right)\sin\left(\frac{\Omega_{eff}t}{2}\right),$$

$$\Omega_{eff}=\sqrt{\delta^2+\Omega^2}$$

$$c_0(t)=\cos\left(\frac{\Omega_{01}t}{2}\right)-i\left(\frac{\Delta e_{01}}{\Omega_{01}}\right)\sin\left(\frac{\Omega_{01}t}{2}\right),$$

$$d_1(t)=-i\left(\frac{2\beta\langle z\rangle}{\Omega_{01}}\right)\sin\left(\frac{\Omega_{01}t}{2}\right),$$

$$\Omega_{01}=\sqrt{4\beta^2\langle z\rangle^2+\Delta e_{01}^2},$$

$$\langle z\rangle_{ij}=\int\phi_i z\phi_j dz,$$

$$\beta=\frac{z_0\delta\Omega}{\Omega_{eff}^2},$$

$$\Delta e_{01}=e_1-e_0.$$

Here δ is the detuning, Δe_{01} is the energy difference between the two trap states, namely, the ground state and the first excited dipole state. This energy difference is held fixed through the derivation, while in actuality they will vary with time. z_0 is the displacement between the two condensates and $\langle z\rangle_{ij}$ is the dipole matrix element which couples the ground and excited states of the trap. In this derivation $\langle z\rangle_{ij}$ is held fixed at $\langle z\rangle_{01}$. The higher couplings are weak and hence neglected.

Taking Eqs. (1) and (2) as starting points, we point out that the $|\psi_1\rangle$ and $|\psi_2\rangle$ individually satisfy the following normalized coupled Gross-Pitaeskii (GP) equation in the Thomas-Fermi limit in an isotropic trap:

$$\frac{d\psi_1(t)}{dt}=\frac{1}{i}\left\{\left[2z_0\langle z\rangle+\lambda_1N_1+N_2+\frac{\delta}{\omega_z}\right]\psi_1(t)+\frac{\Omega}{\omega_z}\psi_2(t)\right\}, \tag{3}$$

$$\frac{d\psi_2(t)}{dt}=\frac{1}{i}\left\{\left[-2z_0\langle z\rangle+\lambda_2N_2+N_1-\frac{\delta}{\omega_z}\right]\times\psi_2(t)+\frac{\Omega}{\omega_z}\psi_1(t)\right\}, \tag{4}$$

$$N_T=N_1+N_2,$$

$$a_\perp=\sqrt{\frac{\hbar}{m\omega}},$$

$$a=a_{11}\sim a_{22},$$

$$\lambda_1=\frac{a_{11}}{a_{12}},$$

$$\lambda_2=\frac{a_{22}}{a_{12}}.$$

In writing the above set of coupled equations, time is in units of trap frequency ω_z and the spatial variables are scaled by $a_\perp$. a_{11} and a_{22} are respectively the s wave scattering lengths for the two hyperfine species of the condensates and a_{12} is the interspecies scattering length. The various energy terms are given with respect to the trap energy level $\hbar\omega_z$. Due to a near degeneracy of a_{11} and a_{22} scattering lengths in the case of ^{87}Rb the approximation that they are equal can be safely carried out. The system is characterized then by a single scattering length a and a single $\lambda=\lambda_1\sim\lambda_2$. In deriving these equations, the spatial dependence of the GP wave-functions are integrated out (adiabatic approximation) with respect to the trap wave-functions, namely, $|\phi_0(z)\rangle$ and $|\phi_1(z)\rangle$ and treated as constants. This assumes that the specific changes in the shape of the trap wave function is not playing a major role in the time evolution of the system. This happens when the trap displacement $z_0=0$ is small such that the coupling (in the fast moving frame) between the motional states and the internal states as given by the parameter β is weak. That is, we are in the *linear* regime as given in [17]. This assumption is an approximation over what is actually experimentally seen [11,7] since we are only interested in the dynamics of the fractional population of the hyperfine states.

We proceed to derive the population fractions and the relative phase differences between the two hyperfine states by noting that

$$|1\rangle=\sqrt{N_1(t)}e^{i\varphi_1(t)}, \tag{5}$$

$$|2\rangle=\sqrt{N_2(t)}e^{i\varphi_2(t)}, \tag{6}$$

$$Z = \frac{N_1 - N_2}{N_T}, \tag{7}$$

$$\theta = \varphi_2 - \varphi_1 . \tag{8}$$

By taking appropriate inner products of $|\psi_1\rangle$ and $|\psi_2\rangle$ with $|\phi_0\rangle$ and $|\phi_1\rangle$ and substituting for $|\psi_1\rangle$ and $|\psi_2\rangle$ in the GP equations, the form given in equations (1) and (2), the following rate equations for average values of fractional population $\langle Z\rangle$ and relative phase $\langle\theta\rangle$ in the two hyperfine states can be derived:

$$\langle \dot{Z}\rangle = \left(\frac{\beta^2\langle z\rangle^2}{\Omega_{01}}\sin(\Omega_{01}t)\right)Z, \tag{9}$$

$$\langle\dot{\theta}\rangle = 4\left(\Delta e_{01}\frac{\beta^2\langle z\rangle^2}{\Omega_{01}^2}\sin^2(\Omega_{01}t)\right) \times\left[\frac{1}{\cos^2\left(\frac{\Omega_{01}t}{2}\right)+\left(\frac{\Delta e_{01}}{\Omega_{01}}\right)^2\sin^2\left(\frac{\Omega_{01}t}{2}\right)}\right]. \tag{10}$$

In deriving the above equations the orthonormal relations of trap wave-functions are assumed to be

$$\langle\phi_i|\phi_j\rangle = \delta_{ij} . \tag{11}$$

Equations (10) and (11) are obtained after averaging over the fast time period namely that of Ω in the problem. So these equations do not explicitly contain Ω. An analytical expression for $\langle Z\rangle$ can then be derived.

$$\langle Z\rangle = Z_0 \exp\left(\frac{\beta^2\langle z\rangle^2}{\Omega_{01}^2}[1-\cos\Omega_{01}t]\right). \tag{12}$$

In the limit of small detuning δ ($\delta \ll \Omega$), this is of the form

$$\langle Z\rangle = Z_0\left[1+2\frac{\beta^2\langle z\rangle^2}{\Omega_{01}^2}\sin^2\left(\frac{\Omega_{01}}{2}t\right)\right]. \tag{13}$$

Here Z_0 is the initial value of the population at time $t=0$.

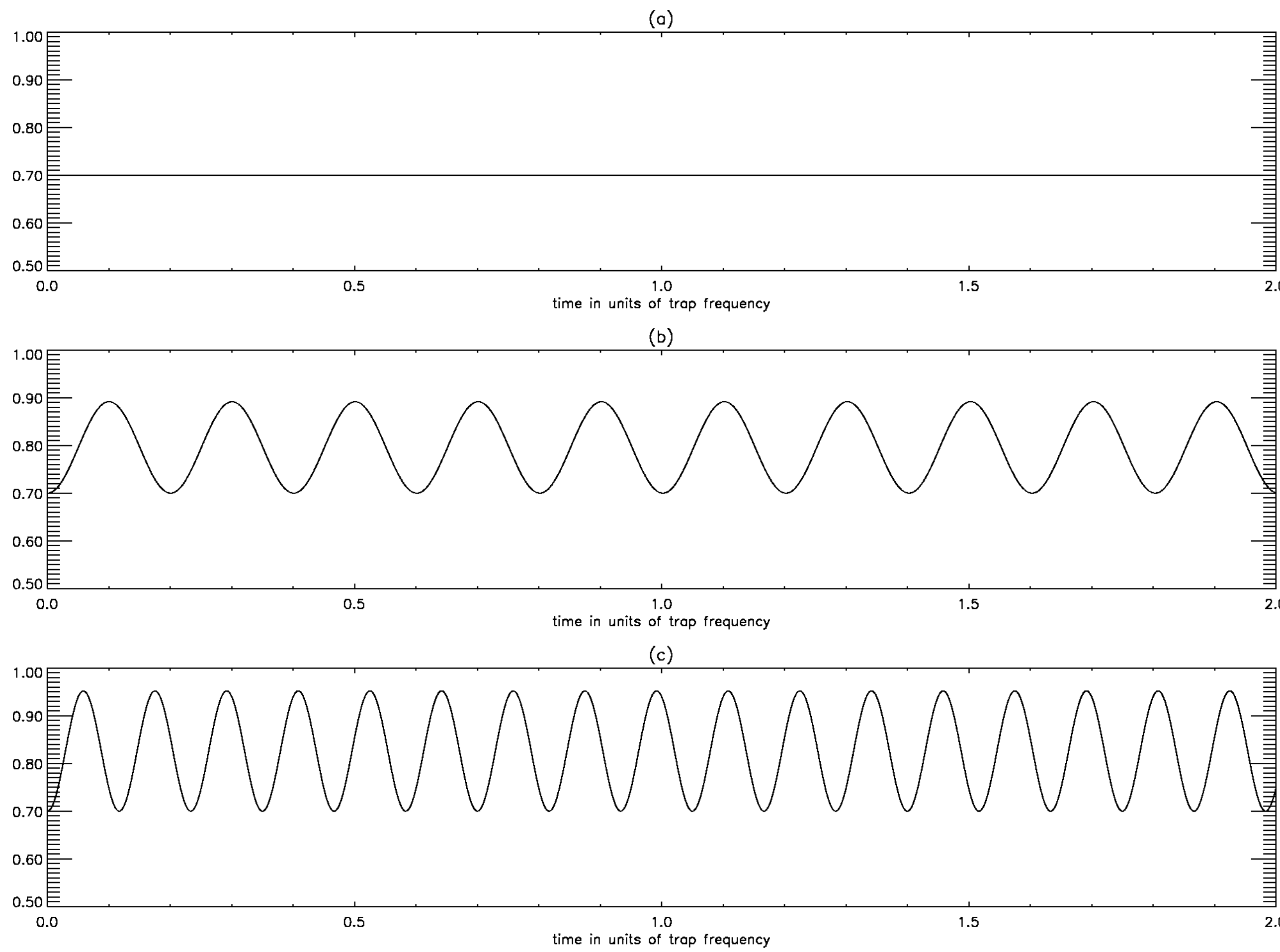

FIG. 1. This plot shows the change in $\langle Z\rangle$ as a function of detuning δ. The parameters are (a) $\delta=0$, (b) $\delta=2\pi\times 50$ Hz, (c) $\delta=2\pi*100$ Hz. The values of other parameters are given in Table I.

TABLE I. Parameter values used to plot Fig. 1 are given here.

N_T	10^5	ν_z	65 Hz
a_{11}	1.0 a_{21}	$a_\perp$	1.3 μm
a_{22}	1.0 a_{21}	a_{21}	5.5 nm

III. RESULTS AND DISCUSSION

Equation (13) has all the essential features which are reported by the experiments and subsequent numerical investigation in [11].

(1) The $\langle Z\rangle$ remains a constant when δ goes to zero or when z_0 goes to zero. That is, in the laboratory frame the fast Rabi oscillations remain unmodulated.

(2) Equation (13) is derived with the implicit assumption that the condensate has a well defined overall phase which can be measured relative to a reference. A decoupling of $\langle\theta\rangle$ and $\langle Z\rangle$ in the time averaged frame over the fast variable occurs as this phase (both the slow and fast varying part) averages to zero.

(3) Though the mean-field term does not explicitly enter the expression (13), we can see that the amplitude of modulation increases with decreasing Δe_{01}, a result which is confirmed by numerical simulations in [11] which predicts a decrease in Δe_{01} for enhanced mean-field effects.

(4) This form of Eq. (13) does not give rise to the chaotic behavior with high values of z_0 reported in [11].

Figure 1 gives a typical curve for the parameters given.

In Ref. [13], a preparation of the vortex mode is presented in this very two-component system. In this contest it is tempting to think of the following scheme for stabilizing such vortices. Starting from the strong coupling regime ($\Omega > \omega_z$), the strength of the external laser field could be gradually decreased over time. At sometime then, when $\Omega < \omega_z$, the population in a particular state (which is a combination of motional and internal states) gets trapped in that state itself, due to macroscopic quantum state trapping (MQST) effects [10], applicable in this regime. More specifically, if such trapping should occur in the first excited motional state, which could be a vortex state, then there seems to be a tremendous improvement achieved in the stability of the vortex.

IV. CONCLUSIONS

In this paper we have analytically derived an expression for the rate of change of fractional population of the hyperfine states $|1,-1\rangle$ and $|2,1\rangle$ of ^{87}Rb in the strong coupling regime using the density-matrix approach. The derivation gives analytical results for population evolution after averaging out the fast dynamical variable, namely, the Rabi period in the problem. This derivation is based on a two-mode model for the trap states as proposed in [11]. The main result of our analytical approach is presented in Eq. (13). This equation reproduces most of the essential features of the two-mode model presented in [11]. Also the possibility to increase the stability of vortex state by modulating Ω is also discussed.

ACKNOWLEDGMENTS

A.N. wishes to acknowledge with pleasure the useful and interesting time spent at ICTP, Trieste, Italy, with Professor S. R. Shenoy and Dr. A. Smerzi for teaching some of the techniques used in these calculations.

[1] E. Goldstein and P. Meystre, Phys. Rev. A **55**, 2935 (1997).
[2] D. Gordon and C.M. Savage, Phys. Rev. A **58**, 1440 (1998) and references therein.
[3] C.J. Myatt *et al.*, Phys. Rev. Lett. **78**, 586 (1997).
[4] H.J. Miesner *et al.*, Phys. Rev. Lett. **82**, 2228 (1999).
[5] J.L. Martin *et al.*, J. Phys. B **32**, 3065 (1999).
[6] M.R. Matthews *et al.*, Phys. Rev. Lett. **81**, 243 (1998).
[7] D.S. Hall *et al.*, Phys. Rev. Lett. **81**, 1539 (1998).
[8] I. Zapata, F. Sols, and A. Legget, Phys. Rev. A **57**, R28 (1998).
[9] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).
[10] S. Raghavan, A. Smerzi, S. Fantoni, and S. Shenoy, Phys. Rev. A **59**, 620 (1999).
[11] J. Williams, R. Walser, J. Cooper, E.A. Cornell, and M. Holland, e-print cond-mat-9904399 (v3).
[12] D.S. Hall *et al.*, Proc. SPIE **3270**, 98 (1998).
[13] M.R. Matthews *et al.*, Phys. Rev. Lett. **83**, 2498 (1999).
[14] F. Dalfovo and S. Stringari, Phys. Rev. A **53**, 2477 (1996).
[15] S. Sinha, Phys. Rev. A **55**, 4325 (1997).
[16] D.S. Rokshar, Phys. Rev. Lett. **79**, 2164 (1997).
[17] A. Sinatra *et al.*, Phys. Rev. Lett. **82**, 251 (1999).