Hadamard regularization of the third post-Newtonian gravitational wave generation of two point masses

Luc Blanchet[1,2,*] and Bala R. Iyer[3,†]

[1]GReCO, Institut d'Astrophysique de Paris (C.N.R.S.), 98bis boulevard Arago, 75014 Paris, France
[2]Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
[3]Raman Research Institute, Bangalore 560080, India

(Received 24 September 2004; published 6 January 2005)

Continuing previous work on the 3PN-accurate gravitational-wave generation from point-particle binaries, we obtain the binary's 3PN mass-type quadrupole and dipole moments for general (not necessarily circular) orbits in harmonic coordinates. The final expressions are given in terms of their core parts, resulting from the application of the pure-Hadamard-Schwartz self-field regularization scheme, and augmented by an ambiguous part. In the case of the 3PN quadrupole we find three ambiguity parameters, ξ, κ and ζ, but only one for the 3PN dipole, in the form of the particular combination $\xi + \kappa$. Requiring that the dipole moment agree with the center-of-mass position deduced from the 3PN equations of motion in harmonic coordinates yields the relation $\xi + \kappa = -9871/9240$. Our results will form the basis of the complete calculation of the 3PN radiation field of compact binaries by means of dimensional regularization.

DOI: 10.1103/PhysRevD.71.024004 PACS numbers: 04.25.−g, 04.30.−w

I. INTRODUCTION

The present paper is the continuation of previous work [1][1] on the generation of gravitational waves by inspiralling compact binaries, viz. binary systems of neutron stars or black holes whose orbit adiabatically spirals in by emission of gravitational radiation. The adiabatic inspiral takes place right before the final plunge and merger of the two compact objects, to (presumably) form a single black hole which will settle down, after emission of its quasinormal modes, into a stationary configuration. Inspiralling compact binaries will almost surely be detected by large scale laser interferometric gravitational-wave observatories like VIRGO and LIGO (Laser Interferometer Gravitational-Wave Observatory). Recent estimates of the rate of coalescences of neutron stars are very promising [2].

It is by now well established (see, e.g., Refs. [3–11]) that, in order to predict in a useful way the gravitational radiation emitted by inspiralling compact binaries, general relativity must be developed to high post-Newtonian (PN) order, probably up to the 3PN or even the 3.5PN level.[2] Correlatively it has been realized that the crucial point for detecting and deciphering the gravitational waves, is to accurately take into account the *gravitational* interaction between the compact objects—responsible for the binary's orbital dynamics and wave emission. Arguments within general relativity [12] show that a good modelization is by *point particles*, characterized only by mass parameters m_1 and m_2 (we neglect the intrinsic spins of the compact objects). It makes sense to implement a model of point particles within the PN approximation, provided that a process of regularization is used for dealing with the infinite self-field of the point particles. The regularization should hopefully be followed by a renormalization.

In paper I we adopted as self-field regularization the Hadamard regularization [13–15], augmented by a prescription for adding a few arbitrary unknown "*ambiguity parameters,*" accounting for the incompleteness of the Hadamard regularization when evaluating certain divergent integrals occurring at the 3PN order. We found that the 3PN mass quadrupole moment of point-particle binaries is complete up to *three* ambiguity parameters, denoted ξ, κ and ζ, which would typically be some rational fractions and could take, within Hadamard's regularization, any numerical values. (The quadrupole moment is the only one to be computed with full 3PN accuracy, thus it contains most of the difficult nonlinear integrals, and all the ambiguities associated with the Hadamard regularization.) The gravitational-wave flux, which is a crucial quantity to be predicted because it drives the binary's orbital phase evolution, has then been found to be complete, in the case of circular orbits, up to a single combination of the latter ambiguity parameters, given by $\theta = \xi + 2\kappa + \zeta$. Of course the ambiguity parameters do not affect the test mass limit of the result of paper I, which is found to be in perfect agreement with the result of linear black-hole perturbations in this limit [5,16–18].

The parameters ξ, κ and ζ represent the analogues, for the case of the gravitational-wave field (more precisely the mass quadrupole moment), of similar parameters which were originally introduced in the problem of the equations of motion of point-particle binaries at the 3PN order [19–22]. More precisely, ξ and κ are the coefficients of some

*Electronic address: blanchet@iap.fr
†Electronic address: bri@rri.res.in
[1]Henceforth Ref. [1] will be referred to as paper I.
[2]As usual the nPN order refers to the terms of order $1/c^{2n}$ in the waveform or energy flux, relative to the lowest-order (Newtonian) approximation described by the Einstein quadrupole formula.

static terms, independent of the particle's velocities but depending on their accelerations, which can be viewed as some analogues of the "static" ambiguity constant ω_s in the 3PN Arnowitt-Deser-Misner (ADM) Hamiltonian [19,20], which is itself equivalent to the parameter λ entering the 3PN equations of motion in harmonic coordinates [21,22]. On the other hand, ζ is the coefficient of a particular velocity-dependent term in the mass quadrupole, and is similar to the "kinetic" ambiguity ω_k in the Hamiltonian [19,20], which has no counterpart in the equations of motion since the velocity terms were unambiguously determined there [21,22]. The work [21,22] used an improved version of the Hadamard regularization called the extended-Hadamard regularization, based on a theory of pseudofunctions and generalized distributional derivatives, and defined in Refs. [23,24].

The ambiguity parameters in the binary's local dynamics (Hamiltonian and/or equations of motion) have been resolved. For the kinetic ambiguity we have $\omega_k = 41/24$, which follows from the requirement of invariance under global Poincaré transformations [21,25]. On the other hand the static ambiguity has been fixed using a powerful argument from dimensional regularization, i.e., computing the binary's dynamics in $d = 3 + \varepsilon$ spatial dimensions and considering the limit where $\varepsilon \to 0$, which led to $\omega_s = 0$ [26] or, equivalently, to $\lambda = -1987/3080$ [27]. The same result has also been achieved in Refs. [28–31] by means of a surface integral approach to the equations of motion of compact objects (i.e., *à la* Einstein-Infeld-Hoffmann), successfully implemented at the 3PN order.

Summarizing, the 3PN equations of motion have been completed in essentially two steps: the first one consists of using Hadamard's regularization and permits the computation of most of the terms but a few; the second step is to apply dimensional regularization in order to fix the value of the few parameters left undetermined. In the present situation it is not possible to compute the equations of motion in the general d-dimensional case, but only in the limit where $\varepsilon = d - 3 \to 0$ [26,27]. In Refs. [26,27] one computes the *difference* between the dimensional and Hadamard regularizations, and it is this "difference," specifically due to the existence of poles in d dimensions (proportional to $1/\varepsilon$), which corresponds to the ambiguities in Hadamard's regularization. Actually the latter difference has to be defined with respect to a particular Hadamard-type regularization of integrals called the "*pure-Hadamard-Schwartz*" (pHS) regularization, following the terminology and definition of Ref. [27]. The pHS regularization consists of the standard notion of Hadamard's partie finie of divergent integrals, together with a minimal treatment of the compact-support (or "contact") terms, and the use of Schwartz distributional derivatives [14]. The result of dimensional regularization is given by the sum of the pHS regularization and of the difference containing the poles proportional to $1/\varepsilon$.

In the present approach, for the 3PN wave generation we basically follow the same two-step strategy as for the 3PN equations of motion, namely:

(i) To obtain the expression of the mass quadrupole moment at 3PN order, as regularized by means of the pHS regularization;

(ii) To add to the pHS result the difference between the dimensional regularization and the pHS one, which as we said above is due to the presence of poles at the 3PN order.

Imposing then that the result of dimensional regularization is *equivalent* to the result of (the pHS variant of) Hadamard's regularization and augmented by appropriate ambiguity parameters, will then uniquely determine the ambiguity parameters. A summary of the above calculations leading to the following unique values for the ambiguity parameters ξ, κ and ζ:

$$\xi = -\frac{9871}{9240}, \qquad \kappa = 0, \qquad \zeta = -\frac{7}{33}, \tag{1.1}$$

has been provided in Ref. [32]. The technical details of the above calculations are now given in a series of three papers of which the present one is the first. Indeed, the present paper is devoted to the calculation of the pHS regularization of the 3PN quadrupole moment, item (i) above. The computation of the part associated with the difference between the dimensional and pHS regularizations cf. item (ii), will be provided in the third paper of this series [33]. In the second paper [34] the value of ζ has been confirmed by a different approach, based on the multipole moments of a boosted Schwarzschild solution.

It should be emphasized that the values (1.1) represent the end result of *dimensional regularization*, obtained by means of the sum of the steps (i) and (ii). However, we shall be able to obtain below [see Eq. (1.2) and Sec. V B] an independent confirmation, within Hadamard's regularization, of the value of the particular combination of parameters $\xi + \kappa$. Moreover, the fact that $\kappa = 0$ has been checked by a diagrammatic reasoning in Ref. [33]. Since as we said ζ has also been computed in Ref. [34] by a different method, we see that the present paper and the works [33,34] altogether provide a check, *independent of dimensional regularization*, for all the parameters (1.1).

In our previous work (paper I), the 3PN mass quadrupole moment was regularized by means of some "hybrid" Hadamard-type regularization, instead of the pHS one, and the ambiguity parameters ξ, κ and ζ were defined with respect to that regularization. In the present paper, since we shall perform a different computation, based on the specific pHS regularization, we shall have to introduce some new ambiguity parameters. Since we do not want to change the definition of ξ, κ and ζ, we shall perform some numerical shifts of the values of ξ, κ and ζ, in order to take into account the different reference points for their definition: hybrid regularization in paper I, vs. pHS regularization in the present paper.

The present investigation will also extend and improve the analysis of paper I in two important ways. First we shall use a better formulation of the multipole moments at the 3PN order, in terms of a set of *retarded* elementary potentials, instead of the "instantaneous" versions of these potentials as was done in paper I. The retarded potentials are the same as in our computation of the equations of motion in harmonic coordinates [22]; their use will appreciably simplify the present work. Secondly we shall generalize paper I to the case of arbitrary orbits, not necessarily circular. Circular orbits are in principle sufficient to describe the inspiralling compact binaries, but the general noncircular case will be mandatory when we want to obtain the values of ξ, κ and ζ separately, and it is also important for a check of the overall consistency of our calculation.

Besides the 3PN mass quadrupole moment of point-particle binaries we compute also their 3PN mass *dipole* moment. The mass dipole is interesting because it is a conserved quantity (or it varies linearly with time), which is already known from the conservative part of the binary's local 3PN equations of motion in harmonic coordinates. Namely the dipole moment is nothing but the integral of the *center-of-mass* position associated with the invariance of the equations of motion under the Poincaré group. It has been computed from the binary's Lagrangian in harmonic coordinates at 3PN order in Refs. [35,36]. In fact what we shall do in the present paper is to *impose* the equivalence between the 3PN dipole moment and the 3PN center-of-mass vector position, and we shall prove that this requirement fixes uniquely *one*, but only one, combination of the ambiguity parameters, viz.

$$\xi + \kappa = -\frac{9871}{9240}, \tag{1.2}$$

working *solely within Hadamard regularization.* This result is perfectly consistent with the complete result provided by dimensional regularization [32] and recalled in Eq. (1.1). We view this agreement as an important check of the correctness of both the present calculation and the one of Ref. [32].

The paper is organized as follows. In Sec. II we review our definitions of the multipole moments of an isolated extended source in the PN approximation, both time-varying moments (having $\ell \geq 2$) and static ones ($\ell = 0, 1$). In Sec. III, we give the explicit expression of the mass-type moments in terms of a set of retarded elementary potentials up to 3PN order. The pHS regularization scheme is then reviewed in Sec. IV, where we comment on the various types of terms encountered in the calculation, and we detail our practical way to perform the partie finie of three-dimensional noncompact-support spatial integrals. Our final results for both the 3PN quadrupole and dipole moments, and our derivation of Eq. (1.2), are presented in Sec. V. The formula for the 3PN quadrupole moment in a general frame turned out to be too long to be published, therefore we choose to present it in the frame of the center-of-mass (but for general orbits): see Eqs. (5.9) and (5.10) below.

II. MULTIPOLE DECOMPOSITION OF THE EXTERIOR FIELD

In this section we provide an account of the relevant notion of multipole moments of a general isolated gravitational-wave source. By definition the moments parametrize the linearized approximation in a post-Minkowskian expansion scheme for the gravitational field in the external (vacuum) domain of the source. Their explicit expressions in terms of the source's physical parameters (matter stress-energy tensor $T^{\mu\nu}$) have been found in the case of a *post-Newtonian* source by using a variant of the theory of matched asymptotic expansions [37–39]. The matching relates the exterior field of the source, as obtained from a multipolar-post-Minkowskian expansion of the external field [40], to the inner field of the post-Newtonian extended source, as iterated in the standard PN way.

A. External solution of the field equations

The Einstein field equations are cast into "relaxed" form by means of the condition of harmonic (or De Donder) coordinates. Denoting the fundamental gravitational field variable by $h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$,[3] this means that

$$\partial_\nu h^{\mu\nu} = 0. \tag{2.1}$$

Under the harmonic-coordinate conditions the field equations take the form of nonlinear wave equations

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}, \tag{2.2}$$

in which $\Box \equiv \eta^{\rho\sigma}\partial_{\rho\sigma}$ denotes the standard flat space-time d'Alembertian operator. The right-hand side of Eq. (2.2) is made of the total (matter plus gravitation) pseudostress-energy tensor in harmonic coordinates given by

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}[h, \partial h, \partial^2 h], \tag{2.3}$$

[3] Here, $g^{\mu\nu}$ denotes the inverse of the usual covariant metric $g_{\mu\nu}$; g is the determinant of $g_{\mu\nu}$: $g = \det(g_{\mu\nu})$; and $\eta^{\mu\nu}$ is an auxiliary Minkowskian metric in Minkowskian coordinates: $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

where $T^{\mu\nu}$ is the stress-energy tensor of the matter fields, and $\Lambda^{\mu\nu}$ represents the gravitational source term which is given by a complicated nonlinear, quadratic at least, functional of $h^{\rho\sigma}$ and its first and second space-time derivatives. Equation (2.3) in paper I gives the explicit expression of $\Lambda^{\mu\nu}$. In the following we shall assume that the support of $T^{\mu\nu}$ is spatially compact. In our formalism the conservation of the pseudotensor,

$$\partial_\nu \tau^{\mu\nu} = 0, \tag{2.4}$$

is the consequence of the harmonic-coordinate condition (2.1).

Let the calligraphic letter $\mathcal{M}$ denote the operation of taking the multipole expansion, so that $\mathcal{M}(h^{\mu\nu})$ represents the multipole expansion of the external gravitational field—a solution of the *vacuum* field equations valid outside the compact support of the matter tensor $T^{\mu\nu}$. Similarly $\mathcal{M}(\Lambda^{\mu\nu})$ denotes the multipole expansion of the gravitational source term, and is obtained from insertion into $\Lambda^{\mu\nu}$ of the multipole expansions of h and its space-time derivatives. Note that $\mathcal{M}(T^{\mu\nu}) = 0$ since $T^{\mu\nu}$ has a compact support. We want to compute the multipole moments of an extended *post-Newtonian* source (one for which the PN approximation is physically meaningful). To this end we first consider the following quantity:

$$\Delta^{\mu\nu} \equiv h^{\mu\nu} - \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathcal{R}}[\tilde{r}^B \mathcal{M}(\Lambda^{\mu\nu})], \tag{2.5}$$

which is made of the difference between $h^{\mu\nu}$, the solution of the field Eqs. (2.2) valid everywhere inside and outside the source, and a particular object obtained from the *retarded* ($\mathcal{R}$) integral of the *multipole* ($\mathcal{M}$) expansion of the gravitational source term $\Lambda^{\mu\nu}$. Here the retarded integral means the usual flat space-time expression

$$\Box^{-1}_{\mathcal{R}} f(\mathbf{x}, t) = -\frac{1}{4\pi}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right). \tag{2.6}$$

Since Eq. (2.6) extends up over the whole space, $\mathbf{x}' \in \mathbb{R}^3$, including the region inside the source where the multipole-moment expansion is not valid (it diverges at the spatial origin $|\mathbf{x}'| \to 0$ located inside the source), one is not allowed to use directly the retarded integral as it stands. This is the reason for the introduction in the second term of Eq. (2.5) of a particular regularization process defined by the *finite part* when a complex number B tends to zero (this operation is abbreviated as $\mathrm{FP}_{B=0}$), and involving the regularization factor

$$\tilde{r}^B \equiv |\tilde{\mathbf{x}}|^B \equiv \left(\frac{r}{r_0}\right)^B, \tag{2.7}$$

which is to be inserted in front of the multipolar-expanded source term of the retarded integral. Here r_0 denotes an arbitrary constant scale having the dimension of a length. Since the divergence of the retarded integral is at the origin of the coordinates, $|\mathbf{x}'| \to 0$ in (2.6), the constant r_0 plays the role of an ultraviolet (UV) cutoff in the second term of Eq. (2.5). However we shall see that in the expression of the multipole moments themselves, given by Eqs. (2.11) or (2.13) below, the *same constant* r_0 will appear to represent in fact an *infrared* (IR) cutoff.

From Eq. (2.5), and noticing that the second term is already in the form of a multipole expansion, we can write the complete multipole decomposition of the external field as

$$\mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Delta^{\mu\nu}) + \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathcal{R}}[\tilde{r}^B \mathcal{M}(\Lambda^{\mu\nu})]. \tag{2.8}$$

This is a solution of the vacuum Einstein field equations,

$$\Box \mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu}), \tag{2.9a}$$

$$\partial_\nu \mathcal{M}(h^{\mu\nu}) = 0, \tag{2.9b}$$

valid in the exterior of the source where $\mathcal{M}(T^{\mu\nu}) = 0$. The first term in Eq. (2.8) is given by an homogeneous solution of the wave equation: $\Box\mathcal{M}(\Delta^{\mu\nu}) = 0$. The second term in (2.8) represents a particular, inhomogeneous, solution, which arises because of the nonlinearities in the external gravitational field, and can be computed by means of the multipolar-post-Minkowskian algorithm of Ref. [40]. In the present paper we shall not consider the second term in (2.8) because its contribution, encompassing many nonlinear effects, has already been computed in Ref. [41] up to 3.5PN order for compact binaries. We shall define our source multipole moments from the contribution $\mathcal{M}(\Delta^{\mu\nu})$, which can be viewed in fact as the "linearized" part of the multipolar decomposition (2.8).

It has been proved in Refs. [37–39] that: (i) the multipole expansion $\mathcal{M}(\Delta^{\mu\nu})$ of the quantity defined by Eq. (2.5) can be computed using the standard formulas (given for instance in Refs. [42,43]) for the multipole expansion outside a *compact-support* source; (ii) the multipole moments admit a very simple expression in the case where the matter source is slowly moving (existence of a small PN parameter $\varepsilon \sim v/c$). The result we find reads as[4]

$$\mathcal{M}(\Delta^{\mu\nu}) = -\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{H}^{\mu\nu}_L\left(t-\frac{r}{c}\right)\right], \tag{2.10}$$

[4]The notation is $L \equiv i_1 \cdots i_\ell$ for a multi-index composed of ℓ multipolar indices $i_1, \cdots, i_\ell$; $\partial_L \equiv \partial_{i_1}\cdots\partial_{i_\ell}$ for the product of ℓ partial derivatives $\partial_i = \partial/\partial x^i$; similarly $x_L \equiv x_{i_1}\cdots x_{i_\ell}$ for the product of ℓ spatial vectors $x^i \equiv x_i$. In the case of summed-up (dummy) multi-indices L, we do not write the ℓ summation symbols, from 1 to 3, over their indices.

where the time-dependent functionals $\mathcal{H}_L^{\mu\nu}$ so introduced, which depend on the retarded time $u = t - r/c$, are explicitly given by[5]

$$\mathcal{H}_L^{\mu\nu}(u) = \mathop{\mathrm{FP}}_{B=0} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B x_L \bar{\tau}^{\mu\nu}(\mathbf{x}, u). \tag{2.11}$$

The integrand of the multipolar functional (2.11) involves the *post-Newtonian* expansion of the total pseudostress-energy tensor given by (2.3), where the formal operation of taking the PN expansion is denoted by means of an overbar, i.e., $\bar{\tau}^{\mu\nu} \equiv \mathrm{PN}[\tau^{\mu\nu}]$. This is the crucial point on which we recognize that the expression (2.11) is valid only for extended PN sources, whose compact support extends well within their own near zone (see [38] for details). The other important feature of Eq. (2.11) is the presence of the finite part operation when $B \to 0$, with regularization factor given by (2.7). The role of the finite part is to deal with the IR divergences initially introduced in the multipole moments by the fact that the PN-expanded integrand of the multipole moments diverges at spatial infinity (when $r \to +\infty$). By contrast, we recall that the finite part in the second term of (2.5) was to take care of the UV divergences when $r \to 0$. The fact that the same finite part operation appears to be either IR or UV depending on the formula is made possible by the properties of the complex analytic continuation (with respect to $B \in \mathbb{C}$). We also mention the fact that the two terms in Eq. (2.8) depend separately on the length scale r_0, but that this dependence is in fact fictitious because the r_0's can be shown to cancel out. [To prove this the best way is to formally differentiate the right-hand side of Eq. (2.8) with respect to r_0.]

B. The STF source multipole moments

In the present approach it is convenient to work with the equivalent of the multipole expansion (2.10) and (2.11) but written in symmetric and trace-free (STF) guise. We present only the results. For the multipole expansion we have

$$\mathcal{M}(\Delta^{\mu\nu}) = -\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \mathcal{F}_L^{\mu\nu}\left(t - \frac{r}{c}\right)\right], \tag{2.12}$$

where the multipole-moment functionals $\mathcal{F}_L^{\mu\nu}(u)$ are now STF with respect to their ℓ indices $L = i_1 \cdots i_\ell$. The $\mathcal{F}_L^{\mu\nu}$'s differ from their counterparts $\mathcal{H}_L^{\mu\nu}$ parametrizing the non-STF multipole decomposition (2.10). They are given by[6]

$$\mathcal{F}_L^{\mu\nu}(u) = \mathop{\mathrm{FP}}_{B=0} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \times \int_{-1}^{1} dz \delta_l(z) \bar{\tau}^{\mu\nu}(\mathbf{y}, u + z|\mathbf{x}|/c). \tag{2.13}$$

Equation (2.13) involves an extra integration, with respect to its non-STF counterpart (2.11), over the variable z, and with associated "weighting" function $\delta_\ell(z)$ given by

$$\delta_\ell(z) \equiv \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1 - z^2)^\ell, \tag{2.14a}$$

$$\int_{-1}^{1} dz \delta_\ell(z) = 1, \tag{2.14b}$$

$$\lim_{\ell \to +\infty} \delta_\ell(z) = \delta(z). \tag{2.14c}$$

Here $\delta(z)$ is Dirac's one-dimensional delta function.

To obtain the *source multipole moments*, we decompose the function $\mathcal{F}_L^{\mu\nu}$, which is already STF in its ℓ indices composing L, into STF-irreducible pieces with respect to all its spatial indices, including those coming from the space-time indices $\mu\nu = \{00, 0i, ij\}$. The appropriate decompositions read[7]

$$\mathcal{F}_L^{00} = R_L, \tag{2.15a}$$

$$\mathcal{F}_L^{0i} = {}^{(+)}T_{iL} + \varepsilon_{ai<i_l}{}^{(0)}T_{L-1>a} + \delta_{i<i_l}{}^{(-)}T_{L-1>}, \tag{2.15b}$$

$$\begin{aligned}\mathcal{F}_L^{ij} = {}&{}^{(+2)}U_{ijL} + \mathop{\mathrm{STF}}_L \mathop{\mathrm{STF}}_{ij}[\varepsilon_{aii_l}{}^{(+1)}U_{ajL-1} \\ &+ \delta_{ii_l}{}^{(0)}U_{jL-1} + \delta_{ii_l}\varepsilon_{aji_{l-1}}{}^{(-1)}U_{aL-2} \\ &+ \delta_{ii_l}\delta_{ji_{l-1}}{}^{(-2)}U_{L-2}] + \delta_{ij}V_L.\end{aligned} \tag{2.15c}$$

We have introduced ten STF tensors R_L, ${}^{(+)}T_{L+1}$, $\cdots$, ${}^{(-2)}U_{L-2}$, V_L, equivalent to the ten components of the original tensor $\mathcal{F}_L^{\mu\nu}$. Because of the harmonic-gauge condition (2.9b), only six of these tensors are independent, and we are led to a set of six source multipole moments, denoted $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. These moments are defined in such a way [38] that the four last ones, $\{W_L, X_L, Y_L, Z_L\}$, parametrize a mere linearized gauge transformation of the linearized part of the multipolar metric, and consequently do not play a very important role. In practice the moments $\{W_L, \cdots, Z_L\}$ appear only at high PN order, where they can be typically computed with Newtonian precision, so they do not pose computational problems. They have already been taken care of in paper I.

The "main" source multipole moments are the mass-type moment I_L and the current-type one J_L. In Sec. III we shall concentrate our attention on the mass moment I_L with full 3PN accuracy. Having in hand the STF-irreducible decompositions (2.15), we obtain in the generic case where

[5]With a slight abuse of notation the generic source point on which one integrates in (2.11) is denoted by $\mathbf{x}$ which is not the same as the field point appearing in the right-hand side of Eq. (2.10).

[6]We denote the STF projection by means of a hat, $\hat{x}_L \equiv \mathrm{STF}(x_{i_1} \cdots x_{i_\ell})$, or sometimes by means of brackets $\langle\rangle$ surrounding the indices, $\hat{x}_L \equiv x_{\langle L \rangle}$.

[7]We denote by ε_{ijk} the usual Levi-Civita antisymmetric symbol such that $\varepsilon_{123} = +1$.

$\ell \geq 2$ (i.e., for nonconserved, arbitrary *time-varying*, moments):

$$I_L = \frac{1}{c^2}(R_L + 3V_L) - \frac{4}{c^3(\ell+1)}{}^{(-)}\dot{T}_L + \frac{2}{c^4(\ell+1)(\ell+2)}{}^{(-2)}\ddot{U}_L, \tag{2.16a}$$

$$J_L = -\frac{\ell+1}{\ell c}{}^{(0)}T_L + \frac{1}{2\ell c^2}{}^{(-1)}\dot{U}_L, \tag{2.16b}$$

where time derivatives are indicated by dots. To express the results (2.16) in the best way, we introduce the following definitions:

$$\Sigma \equiv \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \quad (\text{where } \overline{\tau}^{ii} \equiv \delta_{ij}\overline{\tau}^{ij}), \tag{2.17a}$$

$$\Sigma_i \equiv \frac{\overline{\tau}^{0i}}{c}, \tag{2.17b}$$

$$\Sigma_{ij} \equiv \overline{\tau}^{ij}. \tag{2.17c}$$

For simplicity's sake we omit the overbar of the $\Sigma_{\mu\nu}$'s to indicate the post-Newtonian expansion, but we do not forget that these quantities are given by, and should be treated as, *formal PN-expanded* expressions. The STF source moments, for multipolarities $\ell \geq 2$, are then given by [38]

$$I_L(u) = \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \int_{-1}^{1} dz\left\{\delta_\ell \hat{x}_L \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\delta_{\ell+1}\hat{x}_{iL}\dot{\Sigma}_i + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}\hat{x}_{ijL}\ddot{\Sigma}_{ij}\right\}(\mathbf{x}, u + z|\mathbf{x}|/c), \tag{2.18a}$$

$$J_L(u) = \underset{B=0}{\mathrm{FP}}\, \varepsilon_{ab<i_\ell} \int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \int_{-1}^{1} dz\left\{\delta_\ell \hat{x}_{L-1>a}\Sigma_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\delta_{\ell+1}\hat{x}_{L-1>ac}\dot{\Sigma}_{bc}\right\}(\mathbf{x}, u + z|\mathbf{x}|/c), \tag{2.18b}$$

where the $\Sigma_{\mu\nu}$'s are evaluated at the position $\mathbf{x}$ and at time $u + z|\mathbf{x}|/c$. In the limiting case of *linearized* gravity, we can replace $\overline{\tau}^{\mu\nu}$ by the compact-support matter tensor $T^{\mu\nu}$ and ignore the finite part procedure ($\mathrm{FP}_{B=0}$), so we recover the linearized-gravity expressions obtained in Ref. [43]. Let us emphasize that Eqs. (2.18) are "*exact*," in the sense that they are formally valid up to any PN order. In practice, the PN-expanded moments (2.18) are to be computed by means of the infinite post-Newtonian series

$$\int_{-1}^{1} dz\delta_\ell(z)\Sigma(\mathbf{x}, u + z|\mathbf{x}|/c) = \sum_{k=0}^{+\infty}\alpha_{k,\ell}\left(\frac{|\mathbf{x}|}{c}\frac{\partial}{\partial u}\right)^{2k}\Sigma(\mathbf{x}, u), \tag{2.19a}$$

$$\alpha_{k,\ell} \equiv \frac{(2\ell+1)!!}{(2k)!!(2\ell+2k+1)!!}. \tag{2.19b}$$

In a separate work [34] we shall derive some alternative expressions of the PN moments (2.18) in the form of integrals depending only on the boundary at infinity (i.e., $|\mathbf{x}| \to +\infty$, $u = \text{const}$).

C. The conserved monopole and dipole moments

In the case of the nonradiative moments, i.e., the mass monopole M ($\ell = 0$) and the mass and current dipoles M_i and S_i ($\ell = 1$), things are a little bit more involved than what is given by Eqs. (2.18). The monopole and dipoles are conserved by virtue of the source's equation of motion, Eq. (2.4), namely

$$\dot{M} = 0, \tag{2.20a}$$

$$\ddot{M}_i = 0, \tag{2.20b}$$

$$\dot{S}_i = 0. \tag{2.20c}$$

In particular M denotes the ADM mass of the source. As shown in Ref. [38] the conserved monopole and dipoles can be written into the form

$$M = I + \delta I, \tag{2.21a}$$

$$M_i = I_i + \delta I_i, \tag{2.21b}$$

$$S_i = J_i + \delta J_i, \tag{2.21c}$$

where the first pieces I, I_i and J_i are *defined* by the same formulas as Eqs. (2.18) but in which we set either $\ell = 0$ or $\ell = 1$, and where the extra pieces follow from Eqs. (5.6) in Ref. [38], together with (5.4) and (4.5) there, and are explicitly given by

$$\delta I = \underset{B=0}{\mathrm{FP}}\, B \int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \frac{x_a}{|\mathbf{x}|^2} \times \int_{-1}^{1} dz\left\{-\delta_0\Sigma_a^{(-1)} + \frac{1}{c^2}\delta_1 x_b \Sigma_{ab}\right\}, \tag{2.22a}$$

$$\delta I_i = \underset{B=0}{\mathrm{FP}}\, B \int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \frac{x_a}{|\mathbf{x}|^2} \int_{-1}^{1} dz\left\{-\delta_1 x_i \Sigma_a^{(-1)} - \delta_0\Sigma_{ia}^{(-2)} + \frac{1}{c^2}\delta_2\hat{x}_{ib}\Sigma_{ab}\right\}, \tag{2.22b}$$

$$\delta J_i = \underset{B=0}{\mathrm{FP}}\, B \int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \varepsilon_{iab}\frac{x_{bc}}{|\mathbf{x}|^2}\int_{-1}^{1} dz\delta_1\Sigma_{ac}^{(-1)}. \tag{2.22c}$$

Time antiderivatives are denoted by superscripts $(-n)$; δ_0 and δ_1 refer to the function given by (2.14); like in Eqs. (2.18) the integrands are evaluated at point $\mathbf{x}$ and at time $u + z|\mathbf{x}|/c$. The quantities δI, δI_i and δJ_i are precisely such that the "total" moments M, M_i and S_i obey the conservation laws (2.20) as a consequence of the matter equations of motion (see Ref. [38] for further discussion).

The chief feature of the expressions (2.22) is that they involve an explicit factor B in front, and therefore they depend only on the behavior of the integrand when $|\mathbf{x}| \to +\infty$, since they are zero unless the integral develops a pole $\sim 1/B$ due to the behavior of the integrand near the boundary at infinity. We shall give more details on the way we compute such integrals "at infinity" in Sec. IV D.

III. THE MASS MULTIPOLE MOMENTS AT THE 3PN ORDER

In this section we derive the mass-type source multipole moment I_L (for arbitrary $\ell \geq 2$) at the 3PN approximation, for general extended PN sources. From Eqs. (2.18) we see that one must obtain Σ with full 3PN accuracy, but Σ_i at the 2PN order only, and Σ_{ij} at 1PN order. For this purpose, we make explicit the components of the $\Sigma_{\mu\nu}$'s, defined by Eqs. (2.17), in terms of a certain set of retarded "elementary" potentials: V, V_i, $\hat{W}_{ij}$, $\hat{X}$, $\hat{R}_i$, $\hat{Z}_{ij}$, solutions of appropriate iterated d'Alembertian equations. Although devoid of any direct physical meaning, these potentials have proved to constitute some very useful "building blocks" for practical PN calculations on gravitational-wave generation (paper I), as well as in the problem of equations of motion [22,44]. In paper I we systematically expanded all the retardations in $V, V_i, \cdots$ and introduced some associated "Poisson-like" potentials $U, U_i, \cdots$. Here we shall come back to the same retardedlike potentials $V, V_i, \hat{W}_{ij}, \cdots, \hat{Z}_{ij}$ as in the equations of motion; of course we are motivated by the fact that they have already been computed in Ref. [22]. So we shall redo entirely the computation of paper I, using different elementary potentials and also more systematic MATHEMATICA programs, and in the case of general orbits. Our results will match perfectly with those of paper I.

Let us denote the "matter" parts in the total density contributions (2.17) by

$$\sigma \equiv \frac{T^{00} + T^{ii}}{c^2}; \qquad T^{ii} \equiv \delta_{ij} T^{ij}, \tag{3.1a}$$

$$\sigma_i \equiv \frac{T^{0i}}{c}, \tag{3.1b}$$

$$\sigma_{ij} \equiv T^{ij}. \tag{3.1c}$$

Our chosen definitions for the elementary retarded-type potentials, which involve nonlinear couplings appropriate to 3PN order, are

$$V = \Box_{\mathcal{R}}^{-1}[-4\pi G \sigma], \tag{3.2a}$$

$$V_i = \Box_{\mathcal{R}}^{-1}[-4\pi G \sigma_i], \tag{3.2b}$$

$$\hat{W}_{ij} = \Box_{\mathcal{R}}^{-1}[-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_j V], \tag{3.2c}$$

$$\hat{X} = \Box_{\mathcal{R}}^{-1}[-4\pi G \sigma_{ii} V + \hat{W}_{ij}\partial^2_{ij} V + 2V_i \partial_t \partial_i V + V \partial_t^2 V + \tfrac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i], \tag{3.2d}$$

$$\hat{R}_i = \Box_{\mathcal{R}}^{-1}[-4\pi G(\sigma_i V - \sigma V_i) - 2\partial_k V \partial_i V_k - \tfrac{3}{2}\partial_t V \partial_i V], \tag{3.2e}$$

$$\begin{aligned}\hat{Z}_{ij} = \Box_{\mathcal{R}}^{-1}[&-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk})V - 2\partial_{(i}V\partial_t V_{j)} + \partial_i V_k \partial_j V_k + \partial_k V_i \partial_k V_j - 2\partial_{(i}V_k\partial_k V_{j)} \\ &- \delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k) - \tfrac{3}{4}\delta_{ij}(\partial_t V)^2],\end{aligned} \tag{3.2f}$$

together with the spatial traces denoted by $\hat{W} = \hat{W}_{ii}$ and $\hat{Z} = \hat{Z}_{ii}$. Notice that we shall not need some higher-order potentials, called $\hat{T}$ and $\hat{Y}_i$, which were crucial in the 3PN equations of motion [22]. The 3PN multipole moments are in this sense "less nonlinear" than the 3PN equations of motion.

Like in paper I we find it convenient to decompose I_L into three pieces corresponding to the three terms in (2.18a), respectively, referred to as "scalar" (S), "vector" (V) and "tensor" (T). Applying the formula (2.19) we further decompose each of these pieces into parts, labeled as I, II, III and so on, according to the successive PN contributions. Hence, we write

$$\begin{aligned}I_L = {}&\mathrm{SI}_L + \mathrm{SII}_L + \mathrm{SIII}_L + \mathrm{SIV}_L + \mathrm{VI}_L + \mathrm{VII}_L \\ &+ \mathrm{VIII}_L + \mathrm{TI}_L + \mathrm{TII}_L,\end{aligned} \tag{3.3}$$

in which we consistently neglect all terms that are higher order than 3PN.[8] Without further comment and proof, we give the explicit expressions of all these separate pieces, which are equivalent to the similar expressions given by Eq. (4.2) in paper I. Concerning the "S type,"

[8]We generally do not indicate the PN remainder term $\mathcal{O}(c^{-7})$.

$$
\begin{aligned}
\mathrm{SI}_L = {} & \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \bigg\{ \sigma - \frac{1}{2\pi G c^2}\Delta(V^2) + \frac{4V}{c^4}\sigma_{ii} - \frac{2}{\pi G c^4} V_i \partial_t \partial_i V - \frac{1}{\pi G c^4}\hat{W}_{ij}\partial^2_{ij}V - \frac{1}{2\pi G c^4}(\partial_t V)^2 \\
& + \frac{2}{\pi G c^4}\partial_i V_j \partial_j V_i - \frac{2}{3\pi G c^4}\Delta(V^3) - \frac{1}{2\pi G c^4}\Delta(V\hat{W}) + \frac{16}{c^6}\sigma V_i V_i + \frac{8}{c^6}\sigma_{ii}V^2 + \frac{4}{c^6}\hat{W}_{ij}\sigma_{ij} \\
& + \frac{1}{2\pi G c^6}\hat{W}\partial_t^2 V + \frac{1}{2\pi G c^6} V \partial_t^2 \hat{W} - \frac{2}{\pi G c^6} V(\partial_t V)^2 - \frac{6}{\pi G c^6} V_i \partial_t V \partial_i V - \frac{4}{\pi G c^6} V V_i \partial_t \partial_i V \\
& - \frac{8}{\pi G c^6} V_i \partial_j V_i \partial_j V + \frac{2}{\pi G c^6}(\partial_t V_i)^2 + \frac{1}{\pi G c^6}\partial_t \hat{W} \partial_t V + \frac{4}{\pi G c^6}\partial_i V_j \partial_t \hat{W}_{ij} - \frac{4}{\pi G c^6}\hat{Z}_{ij}\partial_{ij}V \\
& - \frac{4}{\pi G c^6}\partial_t \partial_i V \hat{R}_i + \frac{8}{\pi G c^6}\partial_i V_j \partial_j \hat{R}_i - \frac{2}{3\pi G c^6}\Delta(V^4) - \frac{1}{\pi G c^6}\Delta(V^2\hat{W}) - \frac{1}{4\pi G c^6}\Delta(\hat{W}^2) \\
& + \frac{1}{2\pi G c^6}\Delta(\hat{W}_{ij}\hat{W}_{ij}) - \frac{4}{\pi G c^6}\Delta(V\hat{X}) - \frac{2}{\pi G c^6}\Delta(V\hat{Z}) \bigg\},
\end{aligned} \tag{3.4a}
$$

$$
\begin{aligned}
\mathrm{SII}_L = {} & \frac{1}{2c^2(2\ell+3)} \underset{B=0}{\mathrm{FP}} \frac{d^2}{dt^2} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \bigg\{ |\mathbf{x}|^2 \hat{x}_L \bigg[\sigma + \frac{4V}{c^4}\sigma_{ii} - \frac{2}{\pi G c^4} V_i \partial_t \partial_i V - \frac{1}{\pi G c^4}\hat{W}_{ij}\partial_{ij}V - \frac{1}{2\pi G c^4}(\partial_t V)^2 \\
& + \frac{2}{\pi G c^4}\partial_i V_j \partial_j V_i \bigg] - \frac{2\ell+3}{\pi G c^2}\hat{x}_L V^2 - \frac{1}{2\pi G c^2}\partial_i[\partial_i(V^2)|\mathbf{x}|^2\hat{x}_L - V^2\partial_i(|\mathbf{x}|^2\hat{x}_L)] - \frac{2\ell+3}{\pi G c^4}\hat{x}_L V\hat{W} \\
& - \frac{1}{2\pi G c^4}\partial_i[\partial_i(V\hat{W})|\mathbf{x}|^2\hat{x}_L - V\hat{W}\partial_i(|\mathbf{x}|^2\hat{x}_L)] - \frac{4(2\ell+3)}{3\pi G c^4}\hat{x}_L V^3 - \frac{2}{3\pi G c^4}\partial_i[\partial_i(V^3)|\mathbf{x}|^2\hat{x}_L \\
& - V^3\partial_i(|\mathbf{x}|^2\hat{x}_L)] \bigg\},
\end{aligned} \tag{3.4b}
$$

$$
\begin{aligned}
\mathrm{SIII}_L = {} & \frac{1}{8c^4(2\ell+3)(2\ell+5)} \underset{B=0}{\mathrm{FP}} \frac{d^4}{dt^4} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \bigg\{ |\mathbf{x}|^4 \hat{x}_L \sigma - \frac{2(2\ell+5)}{\pi G c^2}|\mathbf{x}|^2\hat{x}_L V^2 - \frac{1}{2\pi G c^2}\partial_i[\partial_i(V^2)|\mathbf{x}|^4\hat{x}_L \\
& - V^2\partial_i(|\mathbf{x}|^4\hat{x}_L)] \bigg\},
\end{aligned} \tag{3.4c}
$$

$$
\mathrm{SIV}_L = \frac{1}{48c^6(2\ell+3)(2\ell+5)(2\ell+7)} \underset{B=0}{\mathrm{FP}} \frac{d^6}{dt^6} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B |\mathbf{x}|^6 \hat{x}_L \sigma. \tag{3.4d}
$$

Then, the vectorial V parts are

$$
\begin{aligned}
\mathrm{VI}_L = {} & -\frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \underset{B=0}{\mathrm{FP}} \frac{d}{dt} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_{iL} \bigg\{ \sigma_i + \frac{2}{c^2}\sigma_i V - \frac{2}{c^2}\sigma V_i + \frac{1}{\pi G c^2}\partial_j V \partial_i V_j + \frac{3}{4\pi G c^2}\partial_t V \partial_i V \\
& - \frac{1}{2\pi G c^2}\Delta(V V_i) + \frac{2}{c^4}\sigma_i V^2 - \frac{4}{c^4}\sigma \hat{R}_i + \frac{2}{c^4} V_i \sigma_{jj} + \frac{2}{c^4}\hat{W}_{ij}\sigma_j + \frac{2}{c^4} V_j \sigma_{ij} + \frac{1}{2\pi G c^4} V_i \partial_t^2 V - \frac{1}{2\pi G c^4} V \partial_t^2 V_i \\
& + \frac{1}{\pi G c^4}\partial_t V \partial_t V_i - \frac{2}{\pi G c^4} V_j \partial_j \partial_t V_i + \frac{3}{2\pi G c^4} V \partial_t V \partial_i V - \frac{1}{\pi G c^4} V_i \partial_j V \partial_j V + \frac{3}{2\pi G c^4} V_j \partial_i V \partial_j V \\
& + \frac{2}{\pi G c^4}\partial_j V \partial_i \hat{R}_j - \frac{1}{\pi G c^4}\hat{W}_{jk}\partial^2_{jk} V_i + \frac{1}{\pi G c^4}\partial_t \hat{W}_{ij}\partial_j V - \frac{1}{\pi G c^4}\partial_j V_k \partial_i \hat{W}_{jk} + \frac{1}{\pi G c^4}\partial_j \hat{W}_{ik}\partial_k V_j \\
& - \frac{1}{2\pi G c^4}\Delta(V^2 V_i) - \frac{1}{\pi G c^4}\Delta(V\hat{R}_i) - \frac{1}{2\pi G c^4}\Delta(\hat{W} V_i) + \frac{1}{2\pi G c^4}\Delta(\hat{W}_{ij} V_j) \bigg\},
\end{aligned} \tag{3.5a}
$$

$$
\begin{aligned}
\mathrm{VII}_L = {} & -\frac{2(2\ell+1)}{c^4(\ell+1)(2\ell+3)(2\ell+5)} \underset{B=0}{\mathrm{FP}} \frac{d^3}{dt^3} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \bigg\{ |\mathbf{x}|^2 \hat{x}_{iL} \bigg[\sigma_i + \frac{2}{c^2}\sigma_i V - \frac{2}{c^2}\sigma V_i + \frac{1}{\pi G c^2}\partial_j V \partial_i V_j \\
& + \frac{3}{4\pi G c^2}\partial_t V \partial_i V \bigg] - \frac{2\ell+5}{\pi G c^2}\hat{x}_{iL} V V_i - \frac{1}{2\pi G c^2}\partial_j[\partial_j(V V_i)|\mathbf{x}|^2\hat{x}_{iL} - V V_i \partial_j(|\mathbf{x}|^2\hat{x}_{iL})] \bigg\},
\end{aligned} \tag{3.5b}
$$

$$
\mathrm{VIII}_L = -\frac{2\ell+1}{2c^6(\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)} \underset{B=0}{\mathrm{FP}} \frac{d^5}{dt^5} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_{iL} |\mathbf{x}|^4 \sigma_i. \tag{3.5c}
$$

Finally the tensor parts read

$$\mathrm{TI}_L = \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \underset{B=0}{\mathrm{FP}} \frac{d^2}{dt^2} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_{ijL} \left\{ \sigma_{ij} + \frac{1}{4\pi G} \partial_i V \partial_j V + \frac{4}{c^2} \sigma_{ij} V - \frac{4}{c^2} \sigma_i V_j + \frac{2}{\pi G c^2} \partial_i V \partial_t V_j - \frac{1}{\pi G c^2} \partial_i V_k \partial_j V_k + \frac{2}{\pi G c^2} \partial_i V_k \partial_k V_j - \frac{1}{2\pi G c^2} \Delta(V_i V_j) \right\}, \tag{3.6a}$$

$$\mathrm{TII}_L = \frac{2\ell+1}{c^6(\ell+1)(\ell+2)(2\ell+5)(2\ell+7)} \underset{B=0}{\mathrm{FP}} \frac{d^4}{dt^4} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_{ijL} |\mathbf{x}|^2 \left\{ \sigma_{ij} + \frac{1}{4\pi G} \partial_i V \partial_j V \right\}. \tag{3.6b}$$

These formulas *stricto sensu* are valid for general time-varying multipole moments having $\ell \geq 2$. However they constitute also the main contributions in the conserved monopole and dipole moments ($\ell = 0, 1$) as well. In fact we shall prove in Sec. V B that Eqs. (3.4), (3.5), and (3.6) already give the correct answer for the 3PN mass dipole moment of point-particle binaries, i.e., $M_i = I_i$ and the quantity δI_i given in Eq. (2.22b) is zero at 3PN order.

IV. HADAMARD REGULARIZATION OF THE MULTIPOLE MOMENTS

We now specialize the general expression of the 3PN mass moments to compact binary systems modeled by point particles. To this end the first task is to compute all the necessary potentials $\{V, V_i, \hat{W}_{ij}, \cdots\}$, in the case of delta-function singularities, using Hadamard's regularization. Actually the computation of all these potentials has already been done at the occasion of the 3PN equations of motion, and we refer to Ref. [22] for the details. Our next task is to insert these potentials, and their space-time derivatives, into Eqs. (3.3), (3.4), (3.5), and (3.6) for the quadrupole and dipole moments, following the prescriptions of the Hadamard or more precisely the pHS regularization.

A. Pure-Hadamard-Schwartz regularization

Let us first recall the two concepts that constitute the basis of the "ordinary" Hadamard regularization [13,14].[9] The first one concerns the *partie finie of a singular function* at the value of a singular point. The generic function we have to deal with reads $F(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^3$, and becomes singular at the two point-particle singularities located at the positions $\mathbf{y}_1$ and $\mathbf{y}_2$ (in the harmonic-coordinate system). The function $F(\mathbf{x})$ is smooth (C^∞) except at $\mathbf{y}_1$ and $\mathbf{y}_2$, and admits around these singularities some Laurent-type expansions in powers of $r_1 \equiv |\mathbf{x} - \mathbf{y}_1|$ or $r_2 \equiv |\mathbf{x} - \mathbf{y}_2|$. When $r_1 \to 0$ we have, $\forall\ N \in \mathbb{N}$,

$$F(\mathbf{x}) = \sum_{p_0 \leq p \leq N} r_1^p \underset{1}{f}_p(\mathbf{n}_1) + o(r_1^N), \tag{4.1}$$

where the Landau o symbol takes its usual meaning, and the coefficients ${}_1f_p(\mathbf{n}_1)$ are functions of the unit vector $\mathbf{n}_1 \equiv (\mathbf{x} - \mathbf{y}_1)/r_1$.[10] We have $p \in \mathbb{Z}$, bounded from below by some typically negative integer p_0 depending on the F in question. The class of functions such as F is called $\mathcal{F}$; see Ref. [23] for a fuller account of the properties of functions in this class. Now the Hadamard partie finie of F at the singular point $\mathbf{y}_1$, denoted $(F)_1$, is defined by the angular average

$$(F)_1 \equiv \int \frac{d\Omega_1}{4\pi} \underset{1}{f}_0(\mathbf{n}_1), \tag{4.2}$$

where $d\Omega_1 \equiv d\Omega(\mathbf{n}_1)$ is the solid angle element on the unit sphere centered on $\mathbf{y}_1$. Note that the spherical average (4.2) is performed in a *global* inertial frame. In the context of the extended-Hadamard regularization [24] one defines the regularization (4.2) in the Minkowskian "rest frame" of each particle. A distinctive feature of the partie finie (4.2) is its "nondistributivity" in the sense that

$$(FG)_1 \neq (F)_1(G)_1 \text{ in general for } F, G \in \mathcal{F}. \tag{4.3}$$

The second notion in Hadamard's regularization is that of the *partie finie of a divergent integral,* which attributes a value to the integral over $\mathbb{R}^3$ of the function $F(\mathbf{x})$. Consider first two "regularization volumes" around the two singularities $\mathbf{y}_1$ and $\mathbf{y}_2$. We can specifically choose two *spherical* balls (in the considered coordinate system), $B_1(s)$ and $B_2(s)$, centered on the singularities, each of them with radius s. The integral of F over the domain exterior to these balls, i.e., $\mathbb{R}^3 \setminus B_1(s) \cup B_2(s)$, is well defined for any $s > 0$. Hadamard's partie finie (Pf) of the generally divergent integral of F is then *defined* by the always existing limit

$$\mathrm{Pf}_{s_1,s_2} \int d^3\mathbf{x} F(\mathbf{x}) \equiv \lim_{s\to 0} \left\{ \int_{\mathbb{R}^3 \setminus B_1(s) \cup B_2(s)} d^3\mathbf{x} F(\mathbf{x}) + 4\pi \sum_{p+3<0} \frac{s^{p+3}}{p+3} \left(\frac{F}{r_1^p} \right)_1 + 4\pi \ln\left(\frac{s}{s_1} \right) (r_1^3 F)_1 + 1 \leftrightarrow 2 \right\}. \tag{4.4}$$

[9] We refer to Ref. [27] for a digest of possible variants of Hadamard's regularization.

[10] For clearer reading, we use a left-side label 1 like in ${}_1f_p(\mathbf{n}_1)$ when the quantity appears within the text, however the label is always put underneath the quantity when it appears in an equation like in (4.1).

The extra terms, which involve some parties finies in the sense of (4.2), are such that they cancel out the singular part of the "exterior" integral when $s \to 0$. Here the symbol $1 \leftrightarrow 2$ means the same terms but corresponding to the other particle. The two constants s_1 and s_2 entering the logarithmic terms of this definition play a very important role at 3PN order. A way to interpret them is to say that they reflect the arbitrariness in the choice of the regularization volumes surrounding the particles. Indeed it can be checked that the Hadamard partie finie (4.4) does not depend, modulo changing the values of s_1 and s_2, on the *shape* of B_1 and B_2, above chosen as simple spherical balls (see the discussion in Ref. [23]).

The two notions of partie finie, (4.2) and (4.4), are intimately related. Notably the partie-finie integral (4.4) of a gradient is in general nonzero but given by the partie finie, in the sense of (4.2), of some singular function (see [23] for more details). With the definitions (4.2) and (4.4) one can show that, if we want to dispose of a *local* meaning (at any field point $\mathbf{x}$) for the product of F with a delta function, say, $F(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}_1)$, then one cannot simply replace F in front of the delta function by its regularized value. This is a consequence of the nondistributivity of Hadamard's partie finie, Eq. (4.3). Thus,

$$F(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}_1) \neq (F)_1\delta(\mathbf{x}-\mathbf{y}_1) \text{ in general for } F \in \mathcal{F}. \tag{4.5}$$

It is quite evident that the two properties (4.3) and (4.5) are problematic. A remarkable fact is that the problem of the nondistributivity, Eqs. (4.3) and (4.5), arises precisely at the 3PN order, for both the radiation field and the equations of motion, and not before that order. In the problem of the equations of motion we could deal with the properties (4.3) and (4.5) by implementing the extended-Hadamard regularization of Refs. [23,24]. We have not (yet) succeeded in applying the extended-Hadamard regularization to the problem of gravitational-wave generation. For the present paper we choose to follow a different route, and adopt the pHS regularization defined in Ref. [27].

The pHS regularization is a specific "minimal" variant of the Hadamard regularization, which is designed in such a way that it avoids, *by its very definition,* the problematic consequences (4.3) and (4.5) of the ordinary Hadamard regularization. It applies to the case relevant here where the singular function, say, $F_L \in \mathcal{F}$, is made of sums of products of the nonlinear potentials V, V_i, $\hat{W}_{ij}$, $\cdots$ and their space-time derivatives $\partial_i V, \cdots$, and is multiplied by some (regular) multipolar factor $\hat{x}_L$, that is

$$F_L(\mathbf{x}) = \hat{x}_L \mathcal{P}[V, V_i, \hat{W}_{ij}, \cdots, \partial_i V, \cdots], \tag{4.6}$$

where $\mathcal{P}$ denotes a certain multilinear form, i.e., a polynomial in each of its variables $V, V_i, \cdots$. The rules of the pHS regularization are (i) an integral $\int d^3\mathbf{x} F_L(\mathbf{x})$, where F_L takes the form (4.6), is regularized according to the partie-finie prescription (4.4) like in the ordinary Hadamard regularization; (ii) we add the contribution of *distributional* terms coming from the derivatives of potentials, $\partial_i V, \cdots$, according to the usual Schwartz distribution theory [14] (this point is detailed in Sec. IV B); (iii) the regularization of a product of potentials V, V_i, $\hat{W}_{ij}$, $\cdots$ (and their gradients) at a singular point is assumed to be "distributive," which means that the value of F_L at point 1 (say) is given by the replacement rule

$$(F_L)_1 \to \hat{y}_1^L \mathcal{P}[(V)_1, (V_i)_1, (\hat{W}_{ij})_1, \cdots, (\partial_i V)_1, \cdots], \tag{4.7}$$

where the partie finie (4.2) is applied individually on each of the potentials, and $\hat{y}_1^L = \mathrm{STF}(y_1^{i_1}\cdots y_1^{i_\ell})$; and (iv) a contact term, i.e., of the form $F_L(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}_1)$, appearing in the calculation of the sources of the nonlinear potentials and corresponding to their "compact-support" parts, is regularized by means of the rule

$$F_L(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}_1) \to \hat{y}_1^L \mathcal{P}[(V)_1, (V_i)_1, (\hat{W}_{ij})_1, \cdots]\delta(\mathbf{x}-\mathbf{y}_1). \tag{4.8}$$

The rules (4.7) and (4.8) of the pHS regularization are well defined, and are not submitted, by definition, to the unwanted consequences of the nondistributivity of the *ordinary* Hadamard regularization: (4.3) and (4.5). However, as we shall emphasize in Sec. V, the pHS regularization becomes *physically incomplete* at the 3PN order, in the sense that it must be augmented by certain ambiguous contributions, which *a priori* cannot be determined within this regularization scheme.

Our motivation for introducing the pHS regularization is that it constitutes in some sense the *core* of both the Hadamard and dimensional regularizations [27,32]. By "core" we mean that it will yield the complete and correct result for all the terms *but for a few,* and for those which cannot be determined unambiguously the undetermined part will take in general a very special and limited type of structure. Hence the correct result is obtained by adding to the pHS result a limited number of "ambiguous" terms, parametrized by some arbitrary numerical coefficients called ambiguity parameters.

In dimensional regularization the undetermined terms correspond exactly to the contribution of *poles* $\propto 1/\varepsilon$, where $d = 3 + \varepsilon$ is the dimension of space. The complete result in dimensional regularization appears therefore as the sum of the pHS result and what we call the difference, namely, the pole part $\sim 1/\varepsilon$ which can be quite easily obtained from the expansion near the singularities of the functions involved [27,32], and which is nothing but the difference between the dimensional and the pHS regularizations. The method for determining the ambiguity parameters is therefore to equate the ambiguous terms, as they are defined with respect to the pHS regularization, to the latter difference. In the present paper, we compute the pHS regularization of the 3PN binary's quadrupole moment; this constitutes the first and necessary step toward the complete calculation by dimensional regularization (see Ref. [32] for a summary of the method).

B. Schwartz distributional derivatives

We detail here an important feature of the pHS regularization, namely, the systematic use of distributional derivatives *à la* Schwartz [14]. Recall first that previous work on the equations of motion [22] showed that the Schwartz distributional derivatives yield *ill-defined* (formally infinite) terms at the 3PN order in ordinary three-dimensional space. This was a motivation for introducing some appropriate generalized versions of distributional derivatives in the context of the extended-Hadamard regularization [23]. However, one can show [27] that, by working in a space with d dimensions instead of three dimensions, and invoking complex analytic continuation in d, the latter ill-defined terms are in fact rigorously *zero*. The usual Schwartz distributional derivatives are therefore well defined in the context of *dimensional regularization,* and they have been shown to contribute in an essential way to the final equations of motion at 3PN order [27].

In the present paper we include the Schwartz distributional derivatives as part of the calculation based on the pHS regularization. However, as we just pointed out the Schwartz derivatives yield ill-defined terms in three dimensions, so we shall compute them in d spatial dimensions, and then *take the limit*

$$\varepsilon \equiv d - 3 \rightarrow 0. \tag{4.9}$$

This permits us to cancel out (by dimensional continuation) all the formally divergent terms and to perform a perfectly rigorous calculation. Of course, this way of handling the Schwarztian distributional terms shows that in fact the calculation will already constitute a part of a complete calculation using dimensional regularization. However the spirit is different. In the present calculation we use dimensional continuation as a mathematical trick enabling us to give a well-defined meaning to a limited number of terms which would be otherwise infinite. In a real computation based on dimensional regularization the scope is broader, and we should start from the Einstein field equations in d dimensions and consistently perform all the derivations for arbitrary $d \in \mathbb{C}$ before eventually taking the limit (4.9). In the present paper we shall perform our calculation of the Schwartz derivatives based on the expression for the multipole moment given by Eqs. (3.4), (3.5), and (3.6), i.e., without taking into account the modification of the various coefficients which would come from the Einstein field equations in d dimensions. The result in the limit $\varepsilon \rightarrow 0$ will however exactly be the same as by including such d-dependent coefficients because the distributional parts of Schwartz derivatives do not generate any poles proportional to $1/\varepsilon$.

One may ask why is it possible to choose, in the context of Hadamard's regularization, different prescriptions for the distributional derivatives, and nevertheless obtain the same physical result at the end? For instance what would happen if instead of using the Schwartz distributional derivative in the way we have just described, we adopt the generalized derivatives of the extended-Hadamard regularization [23]? The answer which emerges from our detailed computations is that the difference between the final results obtained by different prescriptions takes the form of the ambiguous terms, which are given in the case of the quadrupole moment by the right-hand side of Eq. (5.6) below. Thus, different calculations are in fact equivalent modulo some simple redefinition (or shift) of the values of the ambiguity parameters ξ, κ and ζ.

The d-dimensional calculation of the Schwartz distributional derivatives in the 3PN moments essentially necessitates the same ingredients as in the problem of equations of motion [27]. We introduce some elementary Poisson kernels u_1 and v_1, solving the equations

$$\Delta u_1 = -4\pi\delta^{(d)}(\mathbf{x} - \mathbf{y}_1), \tag{4.10a}$$

$$\Delta v_1 = u_1, \tag{4.10b}$$

where Δ is Laplace's operator in d dimensions and $\delta^{(d)}$ is the Dirac delta function in d dimensions. These kernels play a crucial role in the construction of the d-dimensional versions of the nonlinear potentials [27]. They parametrize the compact-support potential V at 1PN order; evidently u_1 enters the Newtonian part of V while the twice-iterated Poisson kernel v_1 is used for the 1PN retardation. The kernels are given by

$$u_1 = \tilde{k} r_1^{2-d}, \tag{4.11a}$$

$$v_1 = \frac{\tilde{k} r_1^{4-d}}{2(4-d)}, \tag{4.11b}$$

where $r_1 \equiv |\mathbf{x} - \mathbf{y}_1|$ and $\tilde{k}$ is related to the Eulerian Γ function by

$$\tilde{k} = \frac{\Gamma(\frac{d-2}{2})}{\pi^{(d-2)/2}}, \tag{4.12a}$$

$$\lim_{d\rightarrow 3} \tilde{k} = 1. \tag{4.12b}$$

The *second* partial derivative of u_1, and the *fourth* partial derivative of v_1, will contain, besides an ordinary singular function (or pseudofunction) obtained by performing the derivative in an ordinary sense, a distributional component proportional to $\delta^{(d)}$, and given by

$$\partial^2_{ij}(u_1) = \partial^2_{ij}(u_1)_{|_{\text{ordinary}}} - \frac{4\pi}{d}\delta_{ij}\delta^{(d)}(\mathbf{x}-\mathbf{y}_1), \quad (4.13a)$$

$$\partial^4_{ijkl}(v_1) = \partial^4_{ijkl}(v_1)_{|_{\text{ordinary}}} - \frac{4\pi}{d(d+2)}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta^{(d)}(\mathbf{x}-\mathbf{y}_1). \quad (4.13b)$$

These expressions can be derived as particular cases of the Gel'fand-Shilov formula [45]. In addition, we can treat the distributional *time* derivatives in a very simple way from the rule $\partial_t = -v_1^i\partial_i$ applicable to the purely distributional part of the derivative.

The expressions (4.13) permit for instance the computation of the distributional derivative $\partial^2_{ij}V$ at the 1PN level to be inserted into the 1PN source term $\sim\hat{W}_{ij}\partial^2_{ij}V$ in the expression of I_L, cf. Eq. (3.4a). Let us emphasize that the previous method of introducing the Schwartz distributional derivatives in the pHS formalism, i.e., by means of dimensional continuation in d, is probably the only rigorous way to do it. An alternative approach would consist of staying in three dimensions, and employing the generalized derivative operators defined in Ref. [23] (they act on singular functions of the class $\mathcal{F}$ instead of smooth "test" functions with compact support as in Schwartz's distributional theory). But then the result will differ from Schwartz's derivatives by some terms having the structure of the ambiguous terms in the right-hand side of Eq. (5.6).

C. Three-dimensional partie-finie integrals

The main basis of the computation of the 3PN multipole moments of point particles is to perform explicitly many three-dimensional noncompact-support integrals in the sense of the Hadamard partie finie (4.4). In addition to the partie finie we have to take care of the finite part process based on analytic continuation in $B \in \mathbb{C}$ to deal with the boundary of the integrals at infinity. Therefore we must compute explicitly many integrals of the type

$$\mathcal{I}[s_1, s_2, r_0] \equiv \underset{B=0}{\text{FP}}\left\{\text{Pf}_{s_1,s_2}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B F(\mathbf{x})\right\}, \quad (4.14)$$

which depend *a priori* on the two UV-type length scales s_1 and s_2 associated with the Hadamard partie finie (4.4), and on the IR-type length scale r_0 introduced into the general formalism of Sec. II through the regularization factor $|\tilde{\mathbf{x}}|^B \equiv |\mathbf{x}/r_0|^B$.

The function $F(\mathbf{x})$ in Eq. (4.14) stands for a noncompact-support function which, as far as its UV properties are concerned, belongs to the class of singular functions $\mathcal{F}$, i.e., admits some expansions of the type (4.1). The IR behavior of $F(\mathbf{x})$, when $|\mathbf{x}| \to +\infty$, will be specified below. $F(\mathbf{x})$ contains also some multipolar factor such as $\hat{x}_L$ but for simplicity's sake we do not indicate here the multi-index L. In the general case $F(\mathbf{x})$ admits an expression such as Eq. (4.6), i.e., it is given by some multilinear functional of the elementary potentials V, V_i, $\hat{W}_{ij}, \cdots$ and their derivatives. The function F represents the sum of all the *noncompact-support* terms in Eqs. (3.4), (3.5), and (3.6), taking into account only the *ordinary* parts of the derivatives. The compact-support terms in (3.4), (3.5), and (3.6), as well as the purely distributional parts of the Schwartz derivatives [calculated with (4.13)], are treated separately using the rules of the pHS regularization for contact terms, see Eqs. (4.7) and (4.8). On the other hand, we shall point out in Sec. IV D that for many noncompact terms in Eqs. (3.4), (3.5), and (3.6) we can after integrating by parts perform a much simpler computation of these terms, confined to the boundary of the integral "at infinity" and depending on the sole properties of the finite part operation $\text{FP}_{B=0}$.

In this section we explain our practical method for dealing with the integral (4.14). The basic idea is to relate (4.14) to an integral which is convergent at infinity, on which we can thus remove the finite part at $B = 0$, and then to compute this integral by means of the very efficient method of "angular integration" described by Eq. (4.17) in Ref. [23]. We assume for this calculation (this will always be verified in practice) that F admits a powerlike expansion *at infinity*, when $r_1 \to +\infty$ with $t =$ const, of the following type (for any large enough M):

$$F(\mathbf{x}) = \sum_{k_0 \le k \le M} \frac{1}{r_1^k}\,{}_1\varphi_k(\mathbf{n}_1) + o\left(\frac{1}{r_1^M}\right), \quad (4.15)$$

where the coefficients ${}_1\varphi_k$ depend on the unit vector $\mathbf{n}_1 = (\mathbf{x}-\mathbf{y}_1)/r_1$. The index k is bounded from below by some $k_0 \in \mathbb{Z}$. For convenience we have singled out the singularity 1, and considered the expansion when $r_1 \to +\infty$, instead of the more natural choice $|\mathbf{x}| \to +\infty$. Introducing such an asymmetry between the points 1 and 2 is only a matter of convenience, but in fact it is quite appropriate in the present formalism because we shall later use the method of "angular integration" [23] which already particularizes the point 1, around which the angular integration is performed. An advantage is that a good check of the calculation can be done at the end since the final result will have to be symmetric in the particle exchange $1 \leftrightarrow 2$. Next we define an auxiliary function ${}_1F_\infty$ by *subtracting* from F all the terms in its expansion (4.15) which yield some divergencies at infinity, i.e.,

$$\underset{1}{F}{}_\infty(\mathbf{x}) \equiv F(\mathbf{x}) - \sum_{k_0 \le k \le 3} \frac{1}{r_1^k}\,{}_1\varphi_k(\mathbf{n}_1). \quad (4.16)$$

The integral of ${}_1F_\infty$ is easily seen to be convergent at infinity, and therefore it can be computed with the ordinary Hadamard partie-finie prescription given by (4.4). Inserting

(4.16) into (4.14) we obtain

$$\mathcal{I}[s_1, s_2, r_0] = \mathrm{Pf}_{s_1,s_2} \int d^3\mathbf{x} \underset{1}{F}_\infty + \sum_{k_0 \le k \le 3} \underset{B=0}{\mathrm{FP}} \bigg\{ \mathrm{Pf}_{s_1,s_2} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \times \frac{1}{r_1^k} \underset{1}{\varphi}_k(\mathbf{n}_1) \bigg\}. \tag{4.17}$$

We transform the second term in the right-hand side of (4.17). The integrand in the curly brackets is replaced by the following equivalent when $B \to 0$ limited at *first order* in B:

$$\int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \frac{1}{r_1^k} \underset{1}{\varphi}_k(\mathbf{n}_1) = \int d^3\mathbf{r}_1 \tilde{r}_1^B \bigg[1 + B \ln\bigg(\frac{|\mathbf{x}|}{r_1}\bigg) + \mathcal{O}(B^2) \bigg] \frac{1}{r_1^k} \underset{1}{\varphi}_k(\mathbf{n}_1). \tag{4.18}$$

(Notice our change of integration variable, from $\mathbf{x}$ to $\mathbf{r}_1 = \mathbf{x} - \mathbf{y}_1$; we pose $\tilde{r}_1 \equiv r_1/r_0$.) This will turn out to be sufficient for our purpose because at 3PN order one can show that there are no multiple poles in B, therefore the neglected terms $\mathcal{O}(B^2)$ will never contribute at this order. We can prove that the first term in the right-hand side of (4.18) is zero except when $k = 3$, in which case the integral admits a pole, and its finite part depends on the *logarithm* of the ratio between r_0 and s_1. Furthermore, the next term, carrying an explicit factor B, is found to be zero in the case $k = 3$, so we get

$$\underset{B=0}{\mathrm{FP}} \bigg\{ \mathrm{Pf}_{s_1,s_2} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \frac{1}{r_1^3} \underset{1}{\varphi}_3(\mathbf{n}_1) \bigg\} = \ln\bigg(\frac{r_0}{s_1}\bigg) \int d\Omega_1 \underset{1}{\varphi}_3(\mathbf{n}_1). \tag{4.19}$$

Consider next the generic cases where $k \le 2$. It is clear that the integrals are then convergent when $r_1 \to 0$ so we can ignore the Hadamard partie finie (Pf). By analytic continuation in B we find that the first term in (4.18) is now zero, and it remains the next one, which can clearly contribute only if the integral develops a simple pole $\sim 1/B$ at infinity. When $r_1 \to +\infty$ we have the expansion

$$\ln\bigg(\frac{|\mathbf{x}|}{r_1}\bigg) = \frac{1}{2} \sum_{m=1}^{+\infty} \frac{\alpha_m(\mathbf{n}_1)}{r_1^m}, \tag{4.20}$$

where the various coefficients α_m depend on $\mathbf{n}_1$ and also on $\mathbf{y}_1$, and are related to the Gegenbauer polynomial $C_m^\mu(t)$ by[11]

$$\alpha_m(\mathbf{n}_1) = -|\mathbf{y}_1|^m \bigg(\frac{d}{d\mu} \bigg\{ C_m^\mu \bigg[-\frac{(\mathbf{n}_1\mathbf{y}_1)}{|\mathbf{y}_1|} \bigg] \bigg\} \bigg)_{\mu=0}, \tag{4.21}$$

where one sets $\mu = 0$ after differentiation of $C_m^\mu(t)$ with respect to its argument μ. One may want to express (4.21) in a more detailed way with the help of Rodrigues' formula for the Gegenbauer polynomial. Substituting the expansion (4.20) and (4.21) into Eq. (4.18) we are finally in a position to obtain the looked-for result

$$\mathcal{I}[s_1, s_2, r_0] = \mathrm{Pf}_{s_1,s_2} \int d^3\mathbf{x} \underset{1}{F}_\infty + \ln\bigg(\frac{r_0}{s_1}\bigg) \int d\Omega_1 \underset{1}{\varphi}_3(\mathbf{n}_1) - \frac{1}{2} \sum_{m=1}^{+\infty} \int d\Omega_1 \alpha_m(\mathbf{n}_1) \underset{1}{\varphi}_{3-m}(\mathbf{n}_1). \tag{4.22}$$

This formula is systematically employed in our algebraic computer programs. [Of course, the sum in the last term is in fact finite because $3 - m \ge k_0$; see Eq. (4.15).] As we said the first term (partie-finie integral) is computed by means of an "angular integration" around the particle 1 following the procedure defined by (4.17) in Ref. [23]. We have found that the 3PN quadrupole moment resulting from the systematic application of Eq. (4.22) is in perfect agreement with the result of paper I, which was derived by "case-by-case" integration, i.e., using different methods depending on the type and structure of the various terms encountered in the problem.

D. Contributions depending on the boundary at infinity

The result (4.22) can be applied to any of the noncompact-support terms in (3.4), (3.5), and (3.6). However, we now show that many terms can be reexpressed, after suitable integration by parts, in the form of a *surface integral* at infinity $r \equiv |\mathbf{x}| \to +\infty$. Evaluating the surface integral is in general much simpler than performing the "bulk" calculation following Eq. (4.22). The first type of term in (3.4), (3.5), and (3.6) for which a computation "at infinity" is possible takes the form of the finite part ($\mathrm{FP}_{B=0}$) of an integral involving the product of a multipolar STF factor $\hat{x}_L$ with the *Laplacian* of some $G \in \mathcal{F}$, having the structure of a product of elementary potentials, i.e.,

$$\mathcal{J}_L \equiv \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \Delta G. \tag{4.23}$$

There are many such terms in (3.4), (3.5), and (3.6), see for instance the last six terms in Eq. (3.4a). The second type of term which is amenable to a treatment at infinity is composed of the divergence of some vectorial function $H_i \in \mathcal{F}$, containing itself some multipolar factor $\hat{x}_L$ (not indicated in our notation for H_i), say,

[11] Here we follow the standard convention for the Gegenbauer polynomial [46]. In Eq. (4.21) it is calculated at the value $t = -[(\mathbf{n}_1\mathbf{y}_1)/|\mathbf{y}_1|]$, where $(\mathbf{n}_1\mathbf{y}_1)$ denotes the usual scalar product and $|\mathbf{y}_1|$ the usual norm.

$$\mathcal{K} \equiv \underset{B=0}{\text{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \partial_i H_i. \tag{4.24}$$

An example is given by the last two terms in Eq. (3.4b). We deal with these two categories of terms, $\mathcal{J}_L$ and $\mathcal{K}$, in turn.

The Laplacian in $\mathcal{J}_L$ is integrated by parts, and we are allowed to cancel out the all-integrated term which is zero by analytic continuation in $B \in \mathbb{C}$ [because it is zero in the case where $\Re(B)$ is chosen to be a large enough *negative* number], thereby obtaining

$$\begin{aligned}\mathcal{J}_L &= \underset{B=0}{\text{FP}} \int d^3\mathbf{x} \Delta(|\tilde{\mathbf{x}}|^B \hat{x}_L) G \\ &= \underset{B=0}{\text{FP}} B(B + 2\ell + 1) \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^{B-2} \frac{\hat{x}_L}{r_0^2} G.\end{aligned} \tag{4.25}$$

The presence of the factor B means that the result depends only on the polar part $\sim 1/B$ of the integral at the boundary at infinity. Since the pole comes exclusively from a radial integral of the type $\int dr r^{B-1} = r^B/B$, we need only to look for the term of the order of $r^{-\ell-1}$ in the expansion of G when $r \to +\infty$. We compute the expansion of G and obtain

$$G = \cdots + \frac{1}{r^{\ell+1}} X_\ell(\mathbf{n}) + \mathcal{O}\left(\frac{1}{r^{\ell+2}}\right), \tag{4.26}$$

where the dots indicate the terms having different magnitudes in $1/r$ and which thus do not concern us for the present calculation. The interesting coefficient in (4.26) is $X_\ell(\mathbf{n})$, and in terms of it we get

$$\begin{aligned}\mathcal{J}_L &= \underset{B=0}{\text{FP}} B(B + 2\ell + 1) r_0^{-B} \\ &\quad \times \int_{\mathcal{R}}^{+\infty} dr r^{B-1} \int d\Omega \hat{n}_L X_\ell(\mathbf{n}),\end{aligned} \tag{4.27}$$

in which we indicated that the radial integral depends only on a neighborhood of infinity, from some arbitrary radius $\mathcal{R}$ up to $+\infty$. This point is actually not completely obvious at this stage and must be justified in the following way. In the general formalism of Ref. [38], which is valid for extended *smooth* matter distributions, any integral having a factor B in front will depend only on the behavior of the integrand at infinity. Indeed since the matter source is smooth the near zone part of the integral is convergent, thus no poles $\propto 1/B$ can arise due to the UV behavior of the integrand and only the IR-type poles can contribute to the value of the integral. When applying the formalism to point particles one must keep this feature in mind, and replace the stress-energy tensor of an extended source by the $T^{\mu\nu}$ of point particles in the term in question already in the form, by the previous argument, of some far-zone integral. Thus, even for point particles the term depends only on the boundary at infinity and does not explicitly involve UV-type divergencies, although it may implicitly contain some contributions coming from UV divergencies occurring at previous PN iteration steps. Finally, from Eq. (4.27) we readily find the result

$$\mathcal{J}_L = -(2\ell + 1) \int d\Omega \hat{n}_L X_\ell(\mathbf{n}). \tag{4.28}$$

We notice that this result is independent of the arbitrary scale $\mathcal{R}$ introduced in (4.27), as well as of the IR constant r_0. Concerning the integral $\mathcal{K}$ defined by Eq. (4.24) we proceed similarly by integration by parts. We find that the term depends only on the part in the expansion of H_i at infinity which goes like $1/r^2$, hence we look for the coefficient $Y_i(\mathbf{n})$ in

$$H_i = \cdots + \frac{1}{r^2} Y_i(\mathbf{n}) + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{4.29}$$

and we obtain the simple result (independent of $\mathcal{R}$ and r_0)

$$\mathcal{K} = \int d\Omega n_i Y_i(\mathbf{n}). \tag{4.30}$$

In summary, many noncompact-support terms in (3.4), (3.5), and (3.6), having the structure of $\mathcal{J}_L$ and $\mathcal{K}$, are computed by surface integrals at infinity using the properties of the analytic continuation in B. The only task is to look for the relevant coefficients in the expansions of the integrands at infinity, (4.26) or (4.29), and to perform the surface integrals (4.28) or (4.30). This saves a lot of calculations with respect to the bulk calculation of the Hadamard partie finie based on the form found in Eq. (4.22). Of course, the two calculations, at infinity and in the bulk, will completely agree, but notice that for this agreement to work, one must crucially take into account in the bulk calculation, in addition to the formula (4.22), the contribution of the distributional part of derivatives. Thus the Laplacian in Eq. (4.23) is to be considered in a distributional sense.

V. MULTIPOLE MOMENTS OF POINT-PARTICLE BINARIES

A. The 3PN mass quadrupole moment

We have computed the 3PN mass quadrupole moment of point-particle binaries, for general orbits, using the expressions (3.4), (3.5), and (3.6) with $\ell = 2$, following the rules of the pHS regularization, notably the way (4.8) one handles the compact-support contact terms, and the techniques reviewed in Secs. IV C and IV D to compute three-dimensional noncompact-support integrals. We denote by

$I_{ij}^{\text{pHS}}[s_1, s_2, r_0]$ the result of such pHS calculation, in which s_1 and s_2 denote the two UV cutoffs and r_0 the IR length scale. These constants result from the computation of noncompact-support integrals and are shown in our basic formula (4.22) for the Hadamard partie-finie integral. Now it was argued in paper I that the Hadamard regularization of the 3PN quadrupole moment is incomplete, and must be augmented, in order not to be incorrect, by some unknown, ambiguous, contributions.

The first source of ambiguity is the kinetic one, linked to the inability of the Hadamard regularization to ensure the global Poincaré invariance of the formalism (we are speaking here of the ordinary or pHS variants of the Hadamard regularization, as well as of the hybrid regularization which has been used in paper I for the generation problem[12]). As discussed in Sec. X of paper I we must account for the kinetic ambiguity by adding "by hand" a specific ambiguity term, depending on a single ambiguity parameter called ζ. Following here exactly the same reasoning we add to the pHS result the same type of ambiguous term, which means that we must consider as correct the following 3PN quadrupole moment:

$$I_{ij}[s_1, s_2, r_0; \hat{\zeta}] = I_{ij}^{\text{pHS}}[s_1, s_2, r_0] + \frac{44\hat{\zeta}}{3} \frac{G^2 m_1^3}{c^6} v_1^{\langle i} v_1^{j\rangle} + 1 \leftrightarrow 2, \quad (5.1)$$

where the extra term, purely of 3PN order, involves an unknown coefficient $\hat{\zeta}$. Here the coordinate velocity is denoted v_1^i, and the factor $44/3$ is chosen for convenience. The parameter $\hat{\zeta}$ will turn out to be different from the parameter ζ of paper I because we are adding it to the result of the pHS regularization, instead of the hybrid Hadamard-type regularization considered in paper I. [The hybrid regularization differs from the pHS one by the way the contact terms are computed, which takes into account the properties of nondistributivity (4.3) and (4.5) and is more like the one of the extended-Hadamard regularization [23], and in some subtle differences arising between the "case-by-case" computation of the elementary noncompact integrals in paper I and the systematic approach followed here which is based on the formula (4.22). Of course, there is only one thing which is finally important, namely, that these differences are completely encoded into some mere shifts of the values of the ambiguity parameters, see Eqs. (5.7) below.]

The second source of ambiguity is static. It comes from the *a priori* unknown relation between the Hadamard regularization length scales, s_1 and s_2, and the ones, r_1' and r_2', parametrizing the 3PN equations of motion in harmonic coordinates [21,22]. The constants r_1' and r_2' come from the regularization of Poisson-type integrals in the computation of the equations of motion, and can be interpreted as some infinitesimal radial distances used as cutoffs when the field point tends to the singularities. Since we need the equations of motion when computing the time derivatives of the 3PN quadrupole moment, for instance in order to obtain the gravitational-wave flux, we must definitely know the relation between s_1, s_2 and r_1', r_2'. This relation constitutes a true physical undeterminacy within the various variants of Hadamard's regularization (either ordinary, pHS, hybrid or extended). Let us rewrite the right-hand side of Eq. (5.1) by "artificially" introducing r_1' and r_2' into the two slots of the pHS result. For doing this we use the known dependence of the pHS quadrupole in terms of the constants s_1 and s_2. This dependence is the same as in the case of the hybrid quadrupole (and indeed of any other of its regularization variants), and is given by Eq. (10.4) of paper I. Hence we have

$$I_{ij}^{\text{pHS}}[s_1, s_2, r_0] = \frac{44}{3} \frac{G^2 m_1^3}{c^6} \ln\left(\frac{r_{12}}{s_1}\right) y_1^{\langle i} a_1^{j\rangle} + 1 \leftrightarrow 2 + \cdots, \quad (5.2)$$

where a_1^i denotes the Newtonian acceleration and the dots indicate the terms that are independent of s_1 and s_2 (but which can depend on r_0). Using this structure it is evident that the effect of changing $s_1, s_2 \to r_1', r_2'$ in the pHS quadrupole is

$$I_{ij}^{\text{pHS}}[s_1, s_2, r_0] = I_{ij}^{\text{pHS}}[r_1', r_2', r_0] + \frac{44}{3} \frac{G^2 m_1^3}{c^6} \times \ln\left(\frac{r_1'}{s_1}\right) y_1^{\langle i} a_1^{j\rangle} + 1 \leftrightarrow 2. \quad (5.3)$$

We now argue, exactly like in Sec. X of paper I, that the most general admissible structure for the unknown logarithmic ratio in Eq. (5.3) is

$$\ln\left(\frac{r_1'}{s_1}\right) = \hat{\xi} + \hat{\kappa} \frac{m_1 + m_2}{m_1} \quad \text{and} \quad 1 \leftrightarrow 2, \quad (5.4)$$

where $\hat{\xi}$ and $\hat{\kappa}$ denote two new arbitrary ambiguity parameters, which are also *a priori* different from ξ and κ in paper I. The argument leading to Eq. (5.4) is essentially that the quadrupole moment should be a polynomial in the two masses m_1 and m_2 separately. Therefore,

[12] An exception is the extended-Hadamard regularization which is in principle able to preserve the Lorentz invariance, since the Hadamard regularization is performed in the Lorentzian rest frame of each of the particles [24]. However we have not been able to fix the kinetic ambiguity in the 3PN quadrupole moment using the extended-Hadamard regularization.

$$I_{ij}^{\rm pHS}[s_1,s_2,r_0]=I_{ij}^{\rm pHS}[r'_1,r'_2,r_0]+\frac{44}{3}\frac{G^2m_1^3}{c^6}\left(\hat{\xi}+\hat{\kappa}\frac{m_1+m_2}{m_1}\right)\times y_1^{\langle i}a_1^{j\rangle}+1\leftrightarrow 2, \tag{5.5}$$

so we write the Hadamard-regularized 3PN quadrupole, depending on the three ambiguity parameters $\hat{\xi}$, $\hat{\kappa}$ and $\hat{\zeta}$, in the form

$$I_{ij}[s_1,s_2,r_0;\hat{\zeta}]=I_{ij}^{\rm pHS}[r'_1,r'_2,r_0]+\frac{44}{3}\frac{G^2m_1^3}{c^6}\left[\left(\hat{\xi}+\hat{\kappa}\frac{m_1+m_2}{m_1}\right)y_1^{\langle i}a_1^{j\rangle}+\hat{\zeta}v_1^{\langle i}v_1^{j\rangle}\right]+1\leftrightarrow 2. \tag{5.6}$$

We recall from paper I that, by contrast with the latter ambiguity parameters, the three scales r'_1, r'_2 and r_0 are not physical and must disappear from the final results (when they are expressed in a coordinate-invariant way).

Next let us compare the result, for *general noncircular* orbits, with the one of paper I which was obtained by means of the hybrid Hadamard-type regularization.[13] If everything is consistent, Eq. (5.6) should be in perfect agreement with paper I modulo a change of definition of the three ambiguity parameters, due to the use of the pHS regularization here instead of the hybrid regularization in paper I. We find that indeed there is a complete match for all the terms with those of paper I if and only if the ambiguity parameters $\hat{\xi}$, $\hat{\kappa}$ and $\hat{\zeta}$ are related to the corresponding ones ξ, κ and ζ in paper I by

$$\hat{\xi}=\xi+\frac{1}{22}, \tag{5.7a}$$

$$\hat{\kappa}=\kappa, \tag{5.7b}$$

$$\hat{\zeta}=\zeta+\frac{9}{110}. \tag{5.7c}$$

In view of the many differences between the present calculation and the one of paper I (e.g., in the definition of the regularization, the choice of elementary potentials, the way one computes noncompact-support integrals), this agreement constitutes an important check of the lengthy algebra and the correctness of the result. In the following we prefer to come back to the original ambiguity parameters ξ, κ and ζ adopted in paper I, so we write the quadrupole moment as[14]

$$I_{ij}[r'_1,r'_2,r_0;\xi,\kappa,\zeta]=I_{ij}^{\rm pHS}[r'_1,r'_2,r_0]+\frac{44}{3}\frac{G^2m_1^3}{c^6}\left[\left(\xi+\frac{1}{22}+\kappa\frac{m_1+m_2}{m_1}\right)\times y_1^{\langle i}a_1^{j\rangle}+\left(\zeta+\frac{9}{110}\right)v_1^{\langle i}v_1^{j\rangle}\right]+1\leftrightarrow 2. \tag{5.8}$$

Finally we present the result of the computation of all the terms in Eqs. (3.4), (3.5), and (3.6) for $\ell=2$ and general binary orbits. Unfortunately we find that the end expression of the quadrupole is very long in a general frame (with arbitrary origin), so we decide to present only the much shorter expression valid in the frame of the center of mass. The center-of-mass frame is defined by the nullity of the 3PN conserved integral of the center-of-mass vector deduced from the 3PN equations of motion in harmonic coordinates [35]. For this calculation we use the relations between the general and center-of-mass frames given at 3PN order in Ref. [47]. The structure of the 3PN center-of-mass quadrupole moment is[15]

$$I_{ij}[r'_1,r'_2,r_0;\xi,\kappa,\zeta]=\nu m\left\{\left[\mathcal{A}-\frac{24}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r^2}\dot{r}\right]x_{\langle i}x_{j\rangle}+\mathcal{B}\frac{r^2}{c^2}v_{\langle i}v_{j\rangle}+2\left[\mathcal{C}\frac{r\dot{r}}{c^2}+\frac{24}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r}\right]x_{\langle i}v_{j\rangle}\right\}. \tag{5.9}$$

Here we have explicitly displayed the "odd" 2.5PN radiation-reaction contributions. The content of the "even" terms is given by the coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, which generalize to noncircular orbits those given in Eqs. (11.3)–(11.4) of paper I, and read

[13]For the purpose of the comparison we have redone the calculation of paper I in the case of noncircular orbits. In fact, although the end result of paper I is presented for circular orbits, most of the intermediate expressions in this paper are valid for general binary orbits.

[14]We employ the slightly abusive notation that $I_{ij}[s_1,s_2,r_0;\hat{\zeta}]\equiv I_{ij}[r'_1,r'_2,r_0;\xi,\kappa,\zeta]$ when Eqs. (5.4) and (5.7) hold.

[15]Our notation is $m\equiv m_1+m_2$ and $\nu\equiv m_1m_2/m^2$; $x^i\equiv y_1^i-y_2^i$ and $v^i\equiv dx^i/dt=v_1^i-v_2^i$; $v^2=\mathbf{v}^2$ and $\dot{r}=\mathbf{n}.\mathbf{v}$, where $\mathbf{n}\equiv\mathbf{x}/r$ and $r\equiv|\mathbf{x}|$; the STF projection is indicated by brackets surrounding the indices.

$$\mathcal{A} = 1 + \frac{1}{c^2}\left[v^2\left(\frac{29}{42} - \frac{29\nu}{14}\right) + \frac{Gm}{r}\left(-\frac{5}{7} + \frac{8}{7}\nu\right)\right] + \frac{1}{c^4}\left[v^2\frac{Gm}{r}\left(\frac{2021}{756} - \frac{5947}{756}\nu - \frac{4883}{756}\nu^2\right)\right.$$
$$+ \left(\frac{Gm}{r}\right)^2\left(-\frac{355}{252} - \frac{953}{126}\nu + \frac{337}{252}\nu^2\right) + v^4\left(\frac{253}{504} - \frac{1835}{504}\nu + \frac{3545}{504}\nu^2\right) + \left.\dot{r}^2\frac{Gm}{r}\left(-\frac{131}{756} + \frac{907}{756}\nu - \frac{1273}{756}\nu^2\right)\right]$$
$$+ \frac{1}{c^6}\left\{v^6\left(\frac{4561}{11088} - \frac{7993}{1584}\nu + \frac{117067}{5544}\nu^2 - \frac{328663}{11088}\nu^3\right) + v^4\frac{Gm}{r}\left(\frac{307}{77} - \frac{94475}{4158}\nu + \frac{218411}{8316}\nu^2 + \frac{299857}{8316}\nu^3\right)\right.$$
$$+ \left(\frac{Gm}{r}\right)^3\left[\frac{6285233}{207900} + \frac{34091}{1386}\nu - \frac{3632}{693}\nu^2 + \frac{13289}{8316}\nu^3 - \frac{44}{3}\nu(\xi + 2\kappa) - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) - \frac{44}{3}\nu\ln\left(\frac{r}{r'_0}\right)\right]$$
$$+ \dot{r}^2\left(\frac{Gm}{r}\right)^2\left(-\frac{8539}{20790} + \frac{52153}{4158}\nu - \frac{4652}{231}\nu^2 - \frac{54121}{5544}\nu^3\right) + \dot{r}^4\frac{Gm}{r}\left(\frac{2}{99} - \frac{1745}{2772}\nu + \frac{16319}{5544}\nu^2 - \frac{311}{99}\nu^3\right)$$
$$+ v^2\left(\frac{Gm}{r}\right)^2\left(\frac{187183}{83160} - \frac{605419}{16632}\nu + \frac{434909}{16632}\nu^2 - \frac{37369}{2772}\nu^3\right) + v^2\frac{Gm}{r}\dot{r}^2\left(-\frac{757}{5544} + \frac{5545}{8316}\nu - \frac{98311}{16632}\nu^2\right.$$
$$\left.\left. + \frac{153407}{8316}\nu^3\right)\right\}, \tag{5.10a}$$

$$\mathcal{B} = \frac{11}{21} - \frac{11}{7}\nu + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{106}{27} - \frac{335}{189}\nu - \frac{985}{189}\nu^2\right) + v^2\left(\frac{41}{126} - \frac{337}{126}\nu + \frac{733}{126}\nu^2\right) + \dot{r}^2\left(\frac{5}{63} - \frac{25}{63}\nu + \frac{25}{63}\nu^2\right)\right]$$
$$+ \frac{1}{c^4}\left\{v^4\left(\frac{1369}{5544} - \frac{19351}{5544}\nu + \frac{45421}{2772}\nu^2 - \frac{139999}{5544}\nu^3\right) + \left(\frac{Gm}{r}\right)^2\left[-\frac{40716}{1925} - \frac{4294}{2079}\nu + \frac{62576}{2079}\nu^2 - \frac{24314}{2079}\nu^3\right.\right.$$
$$\left. + \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \frac{44}{3}\nu\zeta\right] + \dot{r}^2\frac{Gm}{r}\left(\frac{79}{77} - \frac{5807}{1386}\nu + \frac{515}{1386}\nu^2 + \frac{8245}{693}\nu^3\right) + v^2\frac{Gm}{r}\left(\frac{587}{154} - \frac{67933}{4158}\nu + \frac{25660}{2079}\nu^2\right.$$
$$\left.\left. + \frac{129781}{4158}\nu^3\right) + v^2\dot{r}^2\left(\frac{115}{1386} - \frac{1135}{1386}\nu + \frac{1795}{693}\nu^2 - \frac{3445}{1386}\nu^3\right)\right\}, \tag{5.10b}$$

$$\mathcal{C} = -\frac{2}{7} + \frac{6}{7}\nu + \frac{1}{c^2}\left[v^2\left(-\frac{13}{63} + \frac{101}{63}\nu - \frac{209}{63}\nu^2\right) + \frac{Gm}{r}\left(-\frac{155}{108} + \frac{4057}{756}\nu + \frac{209}{108}\nu^2\right)\right]$$
$$+ \frac{1}{c^4}\left[v^2\frac{Gm}{r}\left(-\frac{2839}{1386} + \frac{237893}{16632}\nu - \frac{188063}{8316}\nu^2 - \frac{58565}{4158}\nu^3\right) + \left(\frac{Gm}{r}\right)^2\left(-\frac{12587}{41580} + \frac{406333}{16632}\nu - \frac{2713}{396}\nu^2\right.\right.$$
$$\left. + \frac{4441}{2772}\nu^3\right) + v^4\left(-\frac{457}{2772} + \frac{6103}{2772}\nu - \frac{13693}{1386}\nu^2 + \frac{40687}{2772}\nu^3\right) + \dot{r}^2\frac{Gm}{r}\left(\frac{305}{5544} + \frac{3233}{5544}\nu - \frac{8611}{5544}\nu^2\right.$$
$$\left.\left. - \frac{895}{154}\nu^3\right)\right]. \tag{5.10c}$$

The 3PN quadrupole moment depends on ξ, κ and ζ, on the constant scale r_0 introduced into the general formalism defined for extended PN sources in Eq. (2.7), and on the "logarithmic barycenter" r'_0 of the two Hadamard self-field regularization scales r'_1 and r'_2, defined by

$$m\ln r'_0 = m_1\ln r'_1 + m_2\ln r'_2. \tag{5.11}$$

Unlike r_0 which cancels out in the complete waveform, already at the level of the general "fluid" formalism, and r'_0 which represents some gauge constant devoid of physical meaning (see [22] and paper I), the ambiguity parameters ξ, κ and ζ represent some genuine physical unknowns, which have recently been computed by means of dimensional regularization in Ref. [32]. We shall show in the next section that it is possible to determine a particular combination of these parameters in the context of Hadamard's regularization.

B. The 3PN mass dipole moment

Recall from Sec. II C that the mass-type dipole moment or "ADM dipole moment" M_i, which varies linearly with time ($\ddot{M}_i = 0$), is the sum of two terms,

$$M_i = I_i + \delta I_i, \tag{5.12}$$

where I_i is defined by the same general expression (2.18a) as for nonconserved moments but in which we set $\ell = 1$, and where the supplementary piece δI_i is given by Eq. (2.22b).

We first concentrate our attention on the first part I_i which is thus given, up to 3PN order, by the explicit expressions (3.4), (3.5), and (3.6) with $\ell = 1$. We follow the same steps as for the quadrupole moment investigated in Sec. V A. Repeating the arguments presented in Sec. X of paper I, we notice first that in the case of the dipole moment there are no ambiguous terms of the kinetic type. Actually one can easily check on dimensional grounds that the existence of such a term in I_i, which would be proportional to v_1^i (plus $1 \leftrightarrow 2$), is impossible. Thus, unlike in the quadrupole case as shown in (5.1), I_i is directly given by the result of the pHS regularization,

$$I_i[s_1, s_2] = I_i^{\mathrm{pHS}}[s_1, s_2]. \tag{5.13}$$

Here s_1 and s_2 are the two regularization scales coming from Eq. (4.4), but as it turns out that there is no dependence on the cutoff scale r_0 in the dipolar case. To define the static ambiguity we must now reexpress the dipole moment in terms of the particular equation-of-motion-related scales r_1' and r_2'. For this we use the dependence of the dipole moment in terms of the scales s_1, s_2,

$$I_i^{\mathrm{pHS}}[s_1, s_2] = \frac{22}{3}\frac{G^2m_1^3}{c^6}a_1^i \ln\left(\frac{r_{12}}{s_1}\right) + 1 \leftrightarrow 2 + \cdots, \tag{5.14}$$

where the dots represent the terms that are independent of s_1 and s_2. Notice the factor 22/3 instead of 44/3 in the quadrupolar case (5.2). This yields immediately

$$I_i^{\mathrm{pHS}}[s_1, s_2] = I_i^{\mathrm{pHS}}[r_1', r_2'] + \frac{22}{3}\frac{G^2m_1^3}{c^6}a_1^i \ln\left(\frac{r_1'}{s_1}\right) + 1 \leftrightarrow 2. \tag{5.15}$$

The ratio r_1'/s_1 is *a priori* unknown but we remember that it has already served for the definition of two of our ambiguity parameters: $\hat{\xi}$ and $\hat{\kappa}$, see Eq. (5.4). Now we shall use in our present calculation of the dipole moment the *same relation* between r_1' and s_1 as was used for the quadrupole moment. This means that we consider that the constants s_1 and s_2 parametrizing the Hadamard partie finie (4.4) have been chosen once and for all at the beginning of both our calculations of the quadrupole and dipole moments, where they take some definite meaning related for instance to the shape of the regularizing volumes B_1 and B_2 which are initially excised around the two singularities when applying Hadamard's definition in the form of Eq. (4.4). Thus we assume that s_1 and s_2 represent some unknown but *fixed* constants—having the same values for the two calculations of the quadrupole and dipole moments.[16] When substituting the expression $\ln(r_1'/s_1) = \hat{\xi} + \hat{\kappa} + \hat{\kappa}m_2/m_1$ into Eq. (5.15) we observe that the last term, which is proportional to the mass ratio m_2/m_1, cancels out after applying the symmetry exchange $1 \leftrightarrow 2$. So we find that the dipole moment depends in fact on *one and only one* combination of ambiguity parameters, namely $\hat{\xi} + \hat{\kappa}$, and we obtain

$$I_i^{\mathrm{pHS}}[s_1, s_2] = I_i^{\mathrm{pHS}}[r_1', r_2'] + \frac{22}{3}\frac{G^2m_1^3}{c^6}(\hat{\xi} + \hat{\kappa})a_1^i + 1 \leftrightarrow 2. \tag{5.16}$$

Then we come back to the original definitions of paper I by using Eqs. (5.7), and this leads to the following expression of the 3PN dipole moment:

$$I_i[r_1', r_2'; \xi + \kappa] = I_i^{\mathrm{pHS}}[r_1', r_2'] + \frac{22}{3}\frac{G^2m_1^3}{c^6}\left(\xi + \kappa + \frac{1}{22}\right)a_1^i + 1 \leftrightarrow 2. \tag{5.17}$$

At this stage we have to worry about the extra contribution δI_i present in Eq. (5.12). From its expression given by (2.22b) we see that obtaining δI_i at the 3PN order requires both Σ_a and Σ_{ab} with the full 3PN accuracy. By contrast, recall that the calculation of I_i necessitated Σ at the 3PN order, but Σ_a and Σ_{ab} with only the 2PN and 1PN precisions, respectively. Thus it seems that δI_i cannot be obtained solely with the formulas developed in Sec. III. Notice that the expression of δI_i involves an explicit factor B, and thus depends only on the presence of IR poles $\propto 1/B$ in the integrals. Consequently δI_i can be computed by the same techniques as in Sec. IV D, i.e., in the form of surface integrals at infinity similar to Eqs. (4.28) or (4.30). We have been able to prove that all the terms in δI_i are *separately zero* up to the 3PN order. For all the terms we did know from using the results of Sec. III we have made a complete calculation, and for the other terms we looked at their allowed *structure* in terms of the basic potentials V, V_i, $\hat{W}_{ij}, \cdots$, invoking dimensionality arguments but leaving aside the unimportant numerical coefficients, which was sufficient to check that the corresponding surface integrals are exactly zero for all the terms. Thus, we conclude that $\delta I_i = 0$ at 3PN order, hence

$$M_i = I_i + \mathcal{O}(c^{-7}), \tag{5.18}$$

which finally results, from the detailed evaluation of all the terms in Eqs. (3.4), (3.5), and (3.6), in[17]

[16]We tried to further extend this type of argument to the constants s_1 and s_2 which were used in the 3PN equations of motion [21,22]. This implied that we had to use for the wave generation the same regularization as in the equations of motion, i.e., the extended-Hadamard regularization [23,24]. Unfortunately this program, whose aim would have been to determine all the ambiguity parameters within Hadamard's regularization (ξ, κ, ζ and also λ), did not fully succeed. Nevertheless a less ambitious part of the program did succeed, and this is what we show here.

[17]The two masses m_1 and m_2 are located at the positions $\mathbf{y}_1$ and $\mathbf{y}_2$, the unit vector between them is $\mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12}$ with $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$, the two coordinate velocities are $\mathbf{v}_1 = d\mathbf{y}_1/dt$ and $\mathbf{v}_2 = d\mathbf{y}_2/dt$, and $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$. Euclidean scalar products are denoted by parenthesis, e.g., $(n_{12}v_1) = \mathbf{n}_{12}.\mathbf{v}_1$.

$$\begin{aligned}
M_i &= m_1 y_1^i + \frac{1}{c^2}\left\{y_1^i\left(-\frac{Gm_1m_2}{2r_{12}} + \frac{m_1v_1^2}{2}\right)\right\} + \frac{1}{c^4}\left\{Gm_1m_2v_1^i\left[-\frac{7}{4}(n_{12}v_1) - \frac{7}{4}(n_{12}v_2)\right] + y_1^i\left(-\frac{5}{4}\frac{G^2m_1^2m_2}{r_{12}^2} + \frac{7}{4}\frac{G^2m_1m_2^2}{r_{12}^2}\right.\right.\\
&\left.\left.+\frac{3}{8}m_1v_1^4 + \frac{Gm_1m_2}{r_{12}}\left[-\frac{1}{8}(n_{12}v_1)^2 - \frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{1}{8}(n_{12}v_2)^2 + \frac{19}{8}v_1^2 - \frac{7}{4}(v_1v_2) - \frac{7}{8}v_2^2\right]\right)\right\}\\
&+\frac{1}{c^6}\left\{v_1^i\left[\frac{235}{24}\frac{G^2m_1^2m_2}{r_{12}}(n_{12}v_{12}) - \frac{235}{24}\frac{G^2m_1m_2^2}{r_{12}}(n_{12}v_{12}) + Gm_1m_2\left(\frac{5}{12}(n_{12}v_1)^3 + \frac{3}{8}(n_{12}v_1)^2(n_{12}v_2)\right.\right.\right.\\
&+\frac{3}{8}(n_{12}v_1)(n_{12}v_2)^2 + \frac{5}{12}(n_{12}v_2)^3 - \frac{15}{8}(n_{12}v_1)v_1^2 - (n_{12}v_2)v_1^2 + \frac{1}{4}(n_{12}v_1)(v_1v_2) + \frac{1}{4}(n_{12}v_2)(v_1v_2)\\
&\left.\left.-(n_{12}v_1)v_2^2 - \frac{15}{8}(n_{12}v_2)v_2^2\right)\right] + y_1^i\left(\frac{5}{16}m_1v_1^6 + \frac{Gm_1m_2}{r_{12}}\left[\frac{1}{16}(n_{12}v_1)^4 + \frac{1}{8}(n_{12}v_1)^3(n_{12}v_2) + \frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2\right.\right.\\
&+\frac{1}{4}(n_{12}v_1)(n_{12}v_2)^3 - \frac{1}{16}(n_{12}v_2)^4 - \frac{5}{16}(n_{12}v_1)^2v_1^2 - \frac{1}{2}(n_{12}v_1)(n_{12}v_2)v_1^2 - \frac{11}{8}(n_{12}v_2)^2v_1^2 + \frac{53}{16}v_1^4\\
&+\frac{3}{8}(n_{12}v_1)^2(v_1v_2) + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2) + \frac{5}{4}(n_{12}v_2)^2(v_1v_2) - 5v_1^2(v_1v_2) + \frac{17}{8}(v_1v_2)^2 - \frac{1}{4}(n_{12}v_1)^2v_2^2\\
&\left.-\frac{5}{8}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{5}{16}(n_{12}v_2)^2v_2^2 + \frac{31}{16}v_1^2v_2^2 - \frac{15}{8}(v_1v_2)v_2^2 - \frac{11}{16}v_2^4\right] + \frac{G^2m_1^2m_2}{r_{12}^2}\left[\frac{79}{12}(n_{12}v_1)^2\right.\\
&\left.-\frac{17}{3}(n_{12}v_1)(n_{12}v_2) + \frac{17}{6}(n_{12}v_2)^2 - \frac{175}{24}v_1^2 + \frac{40}{3}(v_1v_2) - \frac{20}{3}v_2^2\right] + \frac{G^2m_1m_2^2}{r_{12}^2}\left[-\frac{7}{3}(n_{12}v_1)^2 + \frac{29}{12}(n_{12}v_1)(n_{12}v_2)\right.\\
&\left.+\frac{2}{3}(n_{12}v_2)^2 + \frac{101}{12}v_1^2 - \frac{40}{3}(v_1v_2) + \frac{139}{24}v_2^2\right] - \frac{19}{8}\frac{G^3m_1^2m_2^2}{r_{12}^3} + \frac{G^3m_1^3m_2}{r_{12}^3}\left[\frac{55}{18} - \frac{22}{3}(\xi+\kappa) - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right]\\
&\left.\left.+\frac{G^3m_1m_2^3}{r_{12}^3}\left[-\frac{32}{9} + \frac{22}{3}(\xi+\kappa) + \frac{22}{3}\ln\left(\frac{r_{12}}{r'_2}\right)\right]\right)\right\} + 1 \leftrightarrow 2. \qquad (5.19)
\end{aligned}$$

The case of the conserved dipole moment is interesting because it offers us a very good check of the calculations. Indeed let us argue that M_i, which represents the distribution of positions of particles as weighted by their *gravitational masses* m_g, must be *identical* to the position of the center of mass G_i of the system of particles (per unit of total mass), because the center of mass G_i represents in fact the same quantity as the dipole I_i but corresponding to the *inertial masses* m_i of the particles. The equality between mass dipole M_i and center-of-mass position G_i can thus be seen as a consequence of the equivalence principle $m_i = m_g$, which is surely incorporated in our model of point particles. Now the center of mass G_i is already known at the 3PN order for point-particle binaries, as one of the conserved integrals of the 3PN motion in harmonic coordinates (we neglect the radiation-reaction term at 2.5PN order). We recall that G_i, given explicitly in Ref. [35], depends on the regularization length scales r'_1 and r'_2 which are *the same* as in the 3PN equations of motion [21,22] and therefore the same as in our result (5.19)—by *definition* of the ambiguity parameters ξ and κ. However G_i was found in Ref. [35] to be free of ambiguities; for instance the ambiguity parameter λ in the 3PN equations of motion disappears from the expression of G_i. Let us therefore impose the equivalence between M_i and G_i, which means we make the complete identification

$$M_i[r'_1, r'_2; \xi + \kappa] \equiv G_i[r'_1, r'_2], \qquad (5.20)$$

in which we insist that the constants r'_1 and r'_2 appearing in both sides of this equation are the same. Comparing M_i with the expression of G_i given by Eq. (4.5) in Ref. [35], we find that these constants r'_1 and r'_2 cancel out, and that Eq. (5.20) is verified for all the terms *if and only if* the particular combination of ambiguity parameters $\xi + \kappa$ is fixed to the unique value

$$\xi + \kappa = -\frac{9871}{9240}. \qquad (5.21)$$

This result is obtained within Hadamard's regularization. It shows that, although as we have seen Hadamard's regularization is "physically incomplete" (at 3PN order), it can nevertheless be partially completed by invoking some external physical arguments—in the present case the equivalence between mass dipole and center-of-mass position.

More importantly, we find that Eq. (5.21) is nicely consistent with the calculation of the ambiguity parameters by means of dimensional regularization [32,33], whose results have been given in Eq. (1.1). The dimensional regularization *is* complete; it does not need to invoke any "external" physical argument in order to determine the value of all the ambiguity parameters. Nevertheless, it remains that our result (5.21), based simply on a consistency argument (within the overall scheme) between the 3PN equations of motion on the one hand and the 3PN radiation field on the other hand, does provide a verification of the consistency of dimensional regularization itself.

ACKNOWLEDGMENTS

We thank the Indo-French collaboration (IFCPAR) under which this work has been carried out. One of us (L. B.) thanks Misao Sasaki for an invitation at the Yukawa Institute for Theoretical Physics in Kyoto University.

[1] L. Blanchet, B. R. Iyer, and B. Joguet, Phys. Rev. D **65**, 064005 (2002).
[2] V. Kalogera *et al.*, Astrophys. J. **601**, L179 (2004).
[3] C. Cutler *et al.*, Phys. Rev. Lett. **70**, 2984 (1993).
[4] C. Cutler and E. Flanagan, Phys. Rev. D **49**, 2658 (1994).
[5] H. Tagoshi and T. Nakamura, Phys. Rev. D **49**, 4016 (1994).
[6] E. Poisson, Phys. Rev. D **52**, 5719 (1995); **55**, 7980(E) (1997).
[7] T. Damour, B. Iyer, and B. Sathyaprakash, Phys. Rev. D **57**, 885 (1998).
[8] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, Phys. Rev. D **63**, 044023 (2001).
[9] T. Damour, B. R. Iyer, P. Jaranowski, and B. S. Sathyaprakash, Phys. Rev. D **67**, 064028 (2003).
[10] A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D **67**, 024016 (2003).
[11] A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D **67**, 104025 (2003).
[12] T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), pp. 59–144.
[13] J. Hadamard, *Le Problème de Cauchy et les Équations aux Dérivées Partielles Linéaires Hyperboliques* (Hermann, Paris, 1932).
[14] L. Schwartz, *Théorie des Distributions* (Hermann, Paris, 1978).
[15] A. Sellier, Proc. R. Soc. London A **445**, 69 (1994).
[16] M. Sasaki, Prog. Theor. Phys. **92**, 17 (1994).
[17] T. Tanaka, H. Tagoshi, and M. Sasaki, Prog. Theor. Phys. **96**, 1087 (1996).
[18] H. Tagoshi and M. Sasaki, Prog. Theor. Phys. **92**, 745 (1994).
[19] P. Jaranowski and G. Schäfer, Phys. Rev. D **57**, 7274 (1998).
[20] P. Jaranowski and G. Schäfer, Phys. Rev. D **60**, 124003 (1999).
[21] L. Blanchet and G. Faye, Phys. Lett. A **271**, 58 (2000).
[22] L. Blanchet and G. Faye, Phys. Rev. D **63**, 062005 (2001).
[23] L. Blanchet and G. Faye, J. Math. Phys. (N.Y.) **41**, 7675 (2000).
[24] L. Blanchet and G. Faye, J. Math. Phys. (N.Y.) **42**, 4391 (2001).
[25] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **62**, 021501R (2000); **63**, 029903(E) (2001).
[26] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Lett. B **513**, 147 (2001).
[27] L. Blanchet, T. Damour, and G. Esposito-Farèse, Phys. Rev. D **69**, 124007 (2004).
[28] Y. Itoh, T. Futamase, and H. Asada, Phys. Rev. D **62**, 064002 (2000).
[29] Y. Itoh, T. Futamase, and H. Asada, Phys. Rev. D **63**, 064038 (2001).
[30] Y. Itoh and T. Futamase, Phys. Rev. D **68**, 121501 (2003).
[31] Y. Itoh, Phys. Rev. D **69**, 064018 (2004).
[32] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Phys. Rev. Lett. **93**, 091101 (2004).
[33] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer (to be published).
[34] L. Blanchet, T. Damour, and B. R. Iyer, Classical Quantum Gravity **22**, 155 (2005).
[35] V. de Andrade, L. Blanchet, and G. Faye, Classical Quantum Gravity **18**, 753 (2001).
[36] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **63**, 044021 (2001).
[37] L. Blanchet, Phys. Rev. D **51**, 2559 (1995).
[38] L. Blanchet, Classical Quantum Gravity **15**, 1971 (1998).
[39] O. Poujade and L. Blanchet, Phys. Rev. D **65**, 124020 (2002).
[40] L. Blanchet and T. Damour, Philos. Trans. R. Soc. London A **320**, 379 (1986).
[41] L. Blanchet, Classical Quantum Gravity **15**, 113 (1998).
[42] L. Blanchet and T. Damour, Ann. Inst. Henri Poincare Phys. Theor. **50**, 377 (1989).
[43] T. Damour and B. R. Iyer, Phys. Rev. D **43**, 3259 (1991).
[44] L. Blanchet, G. Faye, and B. Ponsot, Phys. Rev. D **58**, 124002 (1998).
[45] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964).
[46] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
[47] L. Blanchet and B. R. Iyer, Classical Quantum Gravity **20**, 755 (2003).