Comment on "Immirzi parameter in quantum general relativity"

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The Immirzi parameter is a free parameter which appears in the physical predictions of loop quantum gravity and is sometimes viewed as a quantization ambiguity. Interpretations have been offered for the Immirzi ambiguity, but there does not appear to be a clear understanding or even a consensus about its origin and significance. We show that a previously discussed example containing a "finite dimensional analogue" of the Immirzi ambiguity is fallacious, in the sense that the ambiguity in this example is not intrinsic to the system, but introduced artificially by compactifying the configuration space.

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A popular approach [1] to the problem of quantizing gravity is loop quantum gravity (LQG). LQG is an offshoot of a program, initiated by Ashtekar [2,3] in the mid 1980s, advocating the use of connection variables rather than metrical variables in canonical gravity. Ashtekar's approach is essentially nonperturbative in spirit and independent of background structures in space-time. Ashtekar's variables consisted of a densitized triad and a complex $SU(2)$ connection. These variables were arrived at by performing a canonical transformation on the extended phase space (EPS) of general relativity. They resulted in a dramatic simplification of the constraints of general relativity. These original Ashtekar variables were complex, which resulted in problems with "reality conditions."

In 1994, Barbero [4] pointed out that a small variation of Ashtekar's canonical transformation resulted in a *real* connection formulation of gravity. The form of the constraints in Barbero's Hamiltonian formulation is not as simple as in Ashtekar's original formulation, but this is a price one is willing to pay to avoid "reality conditions." Current work in LQG is based on the real Hamiltonian formulation due to Barbero [4].

It was pointed out by Immirzi [5] that Barbero's transformation could be slightly generalized. A 1-parameter family (depending on the real parameter β, the "Immirzi parameter") of canonical transformations was possible, all of which had equal claim to validity as Barbero's original transformation. Classically, the Immirzi parameter, which appears as a free parameter in a canonical transformation, has no physical significance and disappears from all physical predictions. However, as Immirzi [5] pointed out, the physical predictions of LQG do depend on the Immirzi parameter. β appears [6] in the spectrum of the area operator and also in the final expression for black hole entropy as calculated from LQG. The β dependence in both these cases is not a "small" correction to a β independent term, but an overall multiplying factor.

Since β appears in the spectrum of the area operator in LQG, it is generally accepted [7–12] that it sets the scale of quantum gravity: the fundamental length in LQG is *not* the Planck length, but $\sqrt{\beta}$ times the Planck length. In [7] an analogy is drawn between the basic unit of electric charge in a loop quantization of Maxwell theory and the fundamental length of LQG. Rovelli and Thiemann [8] criticize and dismiss a number of interpretations which have been offered for the Immirzi ambiguity (IA) and then suggest that it corresponds to a classical canonical transformation that cannot be unitarily implemented. Ashtekar *et al.* [6] compute the entropy of a Schwarzschild black hole in the framework of LQG and find that the entropy is proportional to the area with a constant that depends on the Immirzi parameter. For a particular choice of β, this agrees with the Bekenstein-Hawking entropy of a Schwartzschild black hole. With the *same* choice of β, one also finds agreement for charged black holes. Gambini *et al.* [9] draw parallels between the Immirzi parameter and the θ parameter in QCD. Krasnov [11] offers an argument involving angular momentum bounds on rotating black holes and concludes that the Immirzi parameter must be fixed to unity. This disagrees, however, with the value needed by Ashtekar *et al.* [6] to match the Bekenstein-Hawking value. Rainer [12] attempts to explain between these two distinct values by noticing that the classical limit is tricky. It is clear that some argument is needed to determine the Immirzi parameter so that further predictions of the theory can be made and tested.

The appearance of the unphysical parameter β in physical predictions of LQG is not easy to understand. A theorist is, of course, at liberty to perform any canonical transformation she chooses, but one might have hoped that the physical predictions of the theory would not depend on the whims of the theorist. The fact that the predictions of the theory are affected by the theorist's choice is something that needs to be understood. How does one understand the Immirzi ambiguity? Are there other systems which also display this behavior? The prevailing attitude towards the "Immirzi ambiguity" is that it is a quantization ambiguity. Quantization ambiguities are not unknown in physics, an example being the θ vacua of QCD. The quantum theory contains a new parameter θ, which is absent in the classical theory and has to be fixed by experiment. This quantization ambiguity is now well understood as arising from the multiple connectedness of the configuration space. Similar quantization ambiguities arise for a particle on a circle.

However, the Immirzi ambiguity does not appear to originate in the multiple connectedness of the configuration

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space. In order to fully understand the ambiguity, it would be nice (if possible) to exhibit finite dimensional analogue systems which also suffer from the same ambiguity. Rovelli and Thiemann [8] have made such an effort. They study a number of finite dimensional systems to see if they exhibit the analogue of the Immirzi ambiguity. For the simple harmonic oscillator and for the particle on a circle, they find that there is no ambiguity. They then go on to consider another model which, according to them, does have "Immirzi" ambiguity.

We will show that this example of Rovelli and Thiemann is fallacious: their original system can be quantized without ambiguity. The ambiguity is introduced "by hand" by changing the original configuration space in a β dependent manner. Thus their "quantization prodedure" does not quantize the original system at all, but quantizes a one-parameter family of distinct systems. We now describe the example of Rovelli and Thiemann (in slightly different notation). Consider a free particle in $\mathbb{R}^3$ whose position vector is $\vec{r}$ and whose momentum is $\vec{p}$. The phase space of the system is $\mathbb{R}^6$, the cotangent bundle over $\mathbb{R}^3$. We impose the constraint that the angular momentum of the particle vanishes: $\vec{J} := \vec{r}\times\vec{p}=0$. The Hamiltonian of the system is $H=\vec{p}.\vec{p}/2$. These data define the system and we can proceed to quantize it. Our Hilbert space is the space of square integrable functions on $\mathbb{R}^3$. The constraint $\vec{J}=0$ ensures that the allowed wave functions $\psi(\vec{r})$ are, in fact, spherically symmetric. ψ depends only on $r=|\vec{r}|$ and not on $\vec{r}$. This means that only s waves are allowed. On such states the Hamiltonian is $p_r^2/2$. This describes a particle on the half line $\mathbb{R}^+$ with a measure given by r^2. The Hilbert space is the space of functions $\psi(r)$ which satisfy $\int r^2 dr\, \psi^*(r)\psi(r)<\infty$. With appropriate regularity conditions on $\psi(r)$ at the origin, p_r is a Hermitian operator and the Hamiltonian has a continuous spectrum.

What Rovelli and Thiemann [8] do is to introduce new variables:

$$\vec{A}:=\beta\vec{r}+\frac{\partial f(p)}{\partial\vec{p}} \tag{1}$$

$$\vec{E}:=\vec{p}/\beta \tag{2}$$

using a canonical transformation that depends explicitly on β, the analogue of the Immirzi parameter. Using the analogy with gravity they define g, an element of $SU(2)$, by the formula

$$g:=\exp(i\vec{A}\cdot\vec{\tau}), \tag{3}$$

where $\vec{\tau}$ are the $SU(2)$ generators. They then declare their Hilbert space to be the space of square integrable functions on $SU(2)$, $L^2(SU(2))$ and find that the spectrum of the Hamiltonian is discrete and that it depends on β.

In declaring the Hilbert space to be $L^2(SU(2))$, one has lost contact with the original system [13]. Functions on $SU(2)$ are functions of r with a periodicity of $4\pi/\beta$. Thus, what Rovelli and Thiemann quantize is a system with a different configuration space which has been derived from the orginal one by making identifications in the r coordinate with period $4\pi/\beta$. This leads to a discrete spectrum for the Hamiltonian (since the configuration space is compact), which depends on β (since the size of the configuration space depends on β). However this does not constitute an example of the Immirzi ambiguity: All one has done is to define a new one-parameter family of systems (particles on circles with circumference $4\pi/\beta$) and quantize them. This has nothing to do with quantizing the original system. Thus we conclude that the example given by Rovelli and Thiemann is not a finite dimensional example of the Immirzi ambiguity. The Immirzi ambiguity of Loop quantum gravity is not illuminated by this example.

To summarize, our objection to the example of Rovelli and Thiemann [8] is that the displayed ambiguity is contrived and artificial. The sole motivation for quantizing the simple system with artificially introduced compactifications appears to be to bring out an analogy with the Immirzi ambiguity. Rather than illuminate the Immirzi ambiguity, the example could in fact raise suspicions that the Immirzi ambiguity too is a similarly contrived and artificial phenomenon resulting from arbitrary compactifications on the configuration space.

In recent papers [15,16] it has been suggested that connection based approaches to quantum gravity should learn to deal with non-compact gauge groups. In fact the view advocated in [15,16] would resolve the Immirzi ambiguity in favor of $\pm i$. (These values are unacceptable to Barbero's Hamiltonian formulation and LQG, because the resulting connection variable is not real.) Even if one rejects the view advocated in [15] and works with a real compact gauge group as in LQG, it does seem clear that the Immirzi ambiguity poorly understood. For example, is the Immirzi ambiguity is a genuinely field theoretic phenomenon, needing an infinite number of degrees of freedom to manifest itself? Or do finite dimensional systems exihibit similar behavior? No convincing finite dimensional analogue of the IA has been exihibited to date. If no such examples are found, one may be forced to conclude that the IA is either absent in LQG, or a genuinely field theoretic phenomenon. It would be extremely interesting to settle this question. In another context (QCD and not quantum gravity) there is at least one example where a field theoretic phenomenon (θ vacua) has a simple finite dimensional analogue. Quantization on a multiply connected space (such as a circle) has ambiguities similar to the field theoretic ambiguities. In this case, the finite dimensional example is illuminating because the ambiguity is really there and not just put in "by hand."

From reading the literature one gets the impression that the IA in LQG has been understood in the finite dimensional example of Rovelli and Thiemann [8]. It appears however that the finite dimensional example can be justified only by analogy with the field theoretic situation. Since the field theoretic ambiguity is not understood, there may be a danger of circular reasoning: one justifies the field theoretic ambiguity by using the finite dimensional example and vice versa.

We would like to suggest that the theorist is at liberty to make any canonical transformation she chooses, but it is a reasonable constraint on the calculational scheme she uses that the physical predictions of the theory should not depend on such choices. Indeed, in a recent work by Alexandrov [14] using path integral quantization, the Immirzi ambiguity does not appear, unlike in the LQG approach. It would be interesting to understand why there is such a difference between these approaches.

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[13] In fact, this declaration does considerable violence to the original configuration space. The topology of the original configuration space is changed from $\mathbb{R}$ to S^1. Even metrically, the pullback of the natural measure on $SU(2)$ does not agree with the the natural measure r^2 on $\mathbb{R}^+$. Indeed, the natural measure on $\mathbb{R}^+$ is not even projectable under the map (3) from the original configuration space to $SU(2)$.

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