The most prestigious award for mathematics is the Fields Medal which is awarded once in four years to three or four young mathematicians for their outstanding contributions. They receive the medal during the International Congress of Mathematicians held once in four years.

In the most recent International Congress of Mathematicians held in Berlin, Germany during August 18–August 27, 1998, the following four mathematicians were awarded the Fields Medals: Richard E Borcherds, W Timothy Gowers, Maxim Kontsevich and Curtis T McMullen.

The Work of the Fields Medallists: 1998 [1]

2. William T Gowers

Rajendra Bhatia

The subject Functional Analysis started around the beginning of this century, inspired by a desire to have a unified framework in which the two notions of *continuity* and *linearity* that arise in diverse contexts could be discussed abstractly. The basic objects of study in this subject are Banach spaces and the spaces of bounded (continuous) linear operators on them; the space $C[a, b]$ of continuous functions on an interval $[a, b]$ with the supremum norm, the L^p spaces arising in the theory of integration, the sequence spaces l_p, the Sobolev spaces arising in differential equations, are some of the well-known examples of Banach spaces. Thus there are many concrete examples of the spaces, enabling application of the theory to a variety of problems.

It is generally agreed that finite-dimensional spaces are well understood and thus the main interest lies in infinite-dimensional spaces. A Banach space is *separable* if it has a countable dense subset in it. From now on we will talk only of separable Banach spaces; the nonseparable Banach spaces are too unwieldy.

The simplest examples of infinite-dimensional Banach spaces are the sequence spaces l_p, $1 \leq p < \infty$ consisting of sequences $x = (x_1, x_2, ...)$ for which is finite; the pth root of the latter is taken as the norm of x. These spaces are separable. The space of all bounded sequences, equipped with the supremum norm, is called l_∞. It is not separable, but contains in it the space c_0 consisting of all convergent sequences, which is separable. The following was an open question for a long time: does every Banach space contain in it a subspace that is isomorphic to either c_0 or some l_p, $1 \leq p < \infty$? It was answered in the negative by B. Tsirelson in 1974.

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[1] Reproduced with permission from *Current Science*, Vol.75, pp.1290–1292, 1998.
Part 1. Richard E Borcherds appeared in *Resonance*, Vol.4, No.3, pp.84-87, 1999.

It may be recalled that in the theory of finite-dimensional vector spaces bases play an important role. A *Schauder basis* (or a *topological basis*) for a Banach space X is a sequence $(e_n\}$ in X such that every vector in X has a unique expansion where the infinite series is understood to converge in norm. Unlike in the finite-dimensional case, in general this notion depends on the order in which $\{e_n\}$ is enumerated. We say a Schauder basis $\{e_n\}$ is an *unconditional* basis if $\{ep_{(n)}\}$ is a Schauder basis for every permutation p of natural numbers.

It is easy to see that if a Banach space has a Schauder basis, then it is separable. There was a famous problem as to whether every separable Banach space has a Schauder basis. P Enflo showed in 1973 that the answer is no. It had been shown quite early by S Mazur that every (infinite-dimensional) Banach space has an (infinite-dimensional) subspace with a Schauder basis. (The spaces l_p, $1 \leq p < \infty$ and c_0 do have Schauder bases.)

One of the major results proved by W T Gowers, and independently by B Maurey, in 1991 is that there exist Banach spaces that do not have any infinite-dimensional subspace with an unconditional basis.

In many contexts the interest lies more in operators on a Banach space than the space itself. Many of the everyday examples of Banach spaces do have lots of interesting operators defined on them. But it is not clear whether every Banach space has nontrivial operators acting on it. If the Banach space has a Schauder basis one can construct examples of operators by defining their action on the basis vectors. Shift operators that act by shifting the basis vectors to the left or the right have a very rich structure. Another interesting family of operators is the projections. In a Hilbert space every subspace has an orthogonal complement. So, there are lots of orthogonal decompositions and lots of projections that have infinite rank and corank. In an arbitrary Banach space it is not necessary that any infinite-dimensional subspace must have a complementary subspace. Thus one is not able to construct nontrivial projections in an obvious way.

The construction of Gowers and Maurey was later modified to show that there exists a Banach space X in which every continuous projection has finite rank or corank, and further every subspace of X has the same property. This is equivalent to saying that *no* subspace Y of X can be written as a direct sum $W \oplus Z$ of two infinite-dimensional subspaces. A space with this property is called *hereditarily indecomposable*. In 1993 Gowers and Maurey showed that such a space cannot be isomorphic to *any* of its proper subspaces. This is in striking contrast to the fact that an infinite-dimensional Hilbert space is isomorphic to *each* of its infinite-dimensional subspaces (all of them are isomorphic to l_2). A Banach space with this latter property is called *homogeneous*.

In 1996 Gowers proved a dichotomy theorem showing that every Banach space X contains either a subspace with an unconditional basis or a hereditarily indecomposable subspace. A corollary of this is that every homogeneous space must have an unconditional basis. Combined with another recent result of R Komorowsky and N Tomczak–Jaegermann this leads to another remarkable result: every homogeneous space is isomorphic to l_2.

Another natural question to which Gowers has found a surprising answer is the Schroeder–Bernstein problem for Banach spaces. If X and Y are two Banach spaces, and each is isomorphic to a subspace of the other, then must they be isomorphic? The answer to this question has long been known to be no. A stronger condition on X and Y would be that each is a *complemented* subspace of the other. (A subspace is complemented if there is a continuous projection onto it; we noted earlier that not every subspace has this property.) Gowers has shown that even under this condition, X and Y need not be isomorphic. Furthermore, he showed this by constructing a space Z that is isomorphic to $Z \oplus Z \oplus Z$ but not to $Z \oplus Z$.

All these arcane constructions are not easy to describe. In fact, the norms for these Banach spaces are not given by any explicit formula, they are defined by indirect inductive procedures. All this suggests a potential new development in Functional Analysis. The concept of a Banach space has encompassed many interesting concrete spaces mentioned at the beginning. However, it might be *too* general since it also admits such strange objects. It is being wondered now whether there is a new theory of spaces whose norms are easy to describe. These spaces may have a richer operator theory that general Banach spaces are unable to carry.

In his work Gowers has used techniques from many areas, specially from combinatorics whose methods and concerns are generally far away from those of Functional Analysis. For example, one of his proofs uses the idea of two-person games involving sequences of vectors and Ramsey Theory. Not just that, he has also made several important contributions to combinatorial analysis. We end this summary with an example of such a contribution.

A famous theorem of E. Szemeredi (which solved an old problem of P Erdos) states that for every natural number k and positive real number d there exists N such that every subset of $\{1, 2, \ldots, N\}$ of size d N contains an arithmetic progression of length k. Gowers has found a new proof of this theorem based on Fourier analysis. This proof gives additional important information that the original proof, and some others that followed, could not. It leads to interesting bounds for N in terms of k and d .

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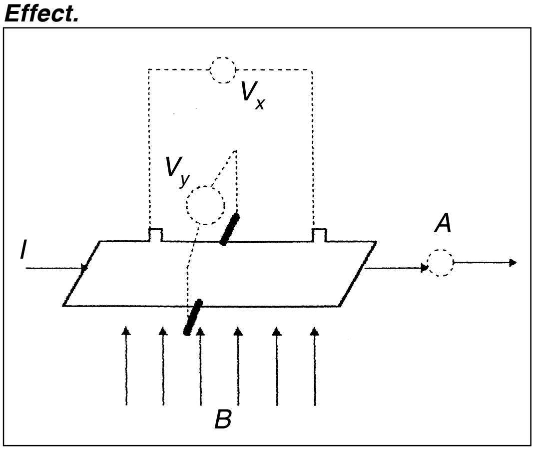
The 1998 Physics Nobel Prize

Electrons Behave as if Split into Three!

R Nityananda

The 1998 Nobel Prize for Physics was awarded to D C Tsui, H L Störmer, and R B Laughlin, all from the USA for the experimental discovery and theoretical understanding of radically new behaviour in a layer of electrons confined by a strong mangetic field at low temperatures. We first go over some background, starting with the work of E Hall in 1879. A magnetic field B was applied normal to a rectangular gold plate, carrying current I along its length *Figure* 1. A 'Hall voltage' V_H was detected across the width of the plate. The simple explanation is that the charge carriers (say electrons) feel a sideways force due to B, but are unable to flow in a circuit in the transverse direction. They therefore accumulate at one edge and build up a potential V_H which cancels the magnetic force $qv_{||} B$. We thus are able to learn the sign of the charge carriers and the velocity with which they move. Notice that for a given current, the velocity of each carrier is *inversely* proportional to the number available per unit area in our layer (*Box* 1). Of course, in Hall's experiments, this number was determined by the properties of the material which were scarcely affected by the weak magnetic fields applied.

Figure 1. Schematic illustration of the Hall Effect.



In the late seventies, Aoki and Ando in Japan realised that the situation could become very different at high magnetic fields and low temperatures in the kind of semiconductor layers used in integrated circuits. The classical orbits are circles. Quantum mechanics replaces the orbit by a wave function. For the lowest state, this wave function gets squeezed into a smaller and smaller area as the field is increased. Since each electron needs its own state, the total number n which can be accomodated in the lowest state goes up proportionally to the magnetic field B, as explained in more detail in *Box* 1.

We have just seen that the ratio $R_H = V_H / I$ (called 'Hall resistance') is proportional to B/n_e. Thus, the expectation was that B would cancel and the Hall resistance would have a universal value h/e^2=25,813Ω. At that time, this result was regarded as a rough approximation. Crystal structure, impurities, and the Coulomb replusion between the electrons were all neglected in the simple model. But very care-ful measurements by von Klitzing in Germany and colleagues in 1980 showed

Box 1.

Let the sample have a unit width, and n particles of charge q per unit area. The current I along the length (longitudinal current) is nqv, where v is the velocity. The Lorentz force qvB has to be balanced by the Hall electric force qV_{H} (for unit width) in the transverse direction. Thus $V_{\mathrm{H}} = vB = IB/nq$, $R_H = VH/I = B/nq$. Thus Hall resistance is proportional to B/n and sensitive to the sign of q as stated in the text.

To understand the B/n ratio for a two dimensional electron gas, we recall that a classical charge q in a field B moving at a speed v describes a circle of radius r, with $mv^2/r = qBv$. The angular frequency $\omega_c = v/r = qB/m$.

The Russian physicist Landau showed that when we apply quantum mechanics to this problem, we get energy levels equally spaced by $\hbar\,\omega_c$, with the lowest at 1/2 ω_c The lowest Landau level corresponds to a classical orbit of energy 1/2 $\omega_c = mv^2/2 = m\omega_c^2\, r^2/2$, hence $r^2 = \quad /m\omega_c = h/2\pi qB$.

We thus see that the area occupied by each orbit is inversely proportional to B. The precise result from Landau's treatment is that the maximum number of electrons which can be accomodated in the lowest level in a unit area = hB/q. One should remember that in the lowest energy state, the electron spin magnetic moment points parallel to the field and the other spin state is higher in energy.

precise integer sub-multiples of this value. The integer was clearly the number of levels filled, but the precision was a great surprise, now exploited all over the world to maintain standards and establish units. Klitzing received the Nobel Prize for Physics in 1985, for discovering this 'Integer Quantum Hall Effect'. The modern theoretical understanding of the IQH is due to Laughlin. His reasoning was based on a beautiful symmetry argument which is however too advanced to describe here. It uses a principle called 'gauge invariance'.

Meanwhile, A C Gossard at Bell Laboratories prepared some of the best samples of another semiconductor. Ga As–GaAl As, in which electrons could be confined to two space dimensions and cooled to very low temperatures in a high magnetic field. D C Tsui and H L Störmer carried out experiments on the Hall effect. They were actually looking for a state of matter conjectured to exist by E P Wigner in 1937, viz electrons avoiding each other and forming a crystalline arrangement. They did not find this. In the course of their experiments, they did find the integer Hall effect but also (for a lower magnetic field) a Hall resistance R_{H} three times larger. It was as if there was a new, stable configuration when the lowest level was just 1/3 filled. Subsequently, other fractions mainly with odd denominators were found (see *Figure* 2 for a modern data set). Again, Laughlin was first off the mark with a theoretical explanation. An essential point in his work is that the Coulomb repulsion between the electrons plays a vital role. Normally, when we describe electrons in solids, we can think of each as a wave in three

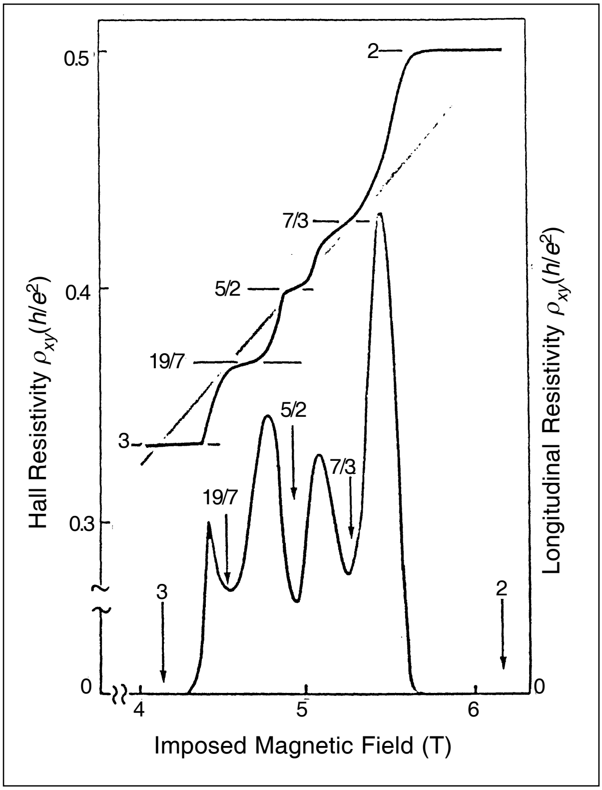


Figure 2. Emergence of the FQH state n= 5/2 at 25mK. The Hall conductivity plateau is seen at 5/2 e^2/h and a pronounced dip in the longitudinal resistivity is also seen at the same magnetic field.

dimensional space, seeing only the average effect of all the others. But in this special situation (FQH), one cannot speak of the wavelike behaviour of one electron without reference to that of the others. One speaks of a 'correlation' between the electrons. Laughlins inspired guess about the nature of this correlation (which is too mathematical to describe at the level of this article) was confirmed by later work. In particular, he showed that the correlations would create excited states in which the total charge in a localised region wa a fraction (like 1/3) of the electronic charge. (The total charge is of course an integer, the difference residing at the boundaries.) This was tested in Israel and France in 1997. Because the electron is the basic unit of charge, even a dc current has fluctuations superposed on it known as 'shot noise'. This is similar to the noise emitted by individual raindrops falling on a roof. The new experiments showed under very special conditions that the FQH state produces shot noise which can be attributed to fractional charge of *e*/3! This strange result was expected from the theory.

Interestingly, the chemistry Nobel Prize for 1998 recognized work on the *quantitative* consequences of electrons being correlated in atoms, molecules, and solids. In the same year, the Physics prize honours work revealing *qualitatively new* behaviour emerging from correlated electron motion. A good earlier example is the phenomenon of superconductivity also earning Nobel Prizes both for the experimental discovery (Kammerlingh Onnes, 1913) and the theory half a century later (Bardeen, Cooper, Schrieffer, 1972).

Acknowledgements

I thank G Baskaran and R Srinivasan for sharing with me drafts of longer articles they have written on the same subject and letting me use material from them.

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