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The quantum measurement problem and selection of classical states*

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Abstract. The problem of selection of preferred basis during passage from quantum to classical systems is treated with the help of a simple example of a 2-state system like the sugar molecule. A simple principle leading to this selection is stated and demonstrated in case of the chosen example. The principle, stated simply is that the preferred basis is the one in which the system environment interaction hamiltonian is diagonal.

Keywords. Quantum measurement; classical limit; quantum watch-dog effect.

1. The problem

I wish to talk about the quantum measurement problem in one of its conventional formulations and point out a new development which, in my view, is a significant step forward towards its solution. I will first state the problem as I see it.

Let us consider a simple two-state quantum system which can be either in state $|+\rangle$ or in $|-\rangle$. Let us also consider an apparatus designed to tell whether the system is in $|+\rangle$ or $|-\rangle$. The apparatus is a macroscopic quantum system which can exist in one of three macroscopically distinct quantum states $|A_0\rangle$, $|A+\rangle$ and $|A-\rangle$, corresponding to let us say, the pointer being in the middle one third, left one third or right one third of the dial. The measurement interaction between system and apparatus establishes the following correlations:

$$|A_0\rangle|+\rangle \xrightarrow{H_{\text{meas.}}} |A+\rangle|+\rangle, \tag{1}$$

and

$$|A_0\rangle|-\rangle \xrightarrow{H_{\text{meas.}}} |A-\rangle|-\rangle, \tag{2}$$

where $|A_0\rangle$ is the initial state of the apparatus. Inspection of the final state of the apparatus then reveals the state of the quantum system. If one always made measurements on systems which are either in $|+\rangle$ or in $|-\rangle$, there would be no measurement problem. The problem arises when one notes that quantum mechanics admits more general states of the form:

$$|\psi\rangle = a|+\rangle + b|-\rangle, \tag{3}$$

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where a and b are complex numbers such that

$$|a|^2+|b|^2=1, \tag{4}$$

when the apparatus described above interacts with a system in state $|\psi\rangle$, it follows from the linear quantum law of time-evolution that one gets the correlation

$$|A_0\rangle|\psi\rangle \xrightarrow{H_{\text{meas.}}} a|A+\rangle|+\rangle+b|A-\rangle|-\rangle. \tag{5}$$

In the language of the density matrix, the state on the right side is represented by the density matrix

$$\rho=\begin{pmatrix}|a|^2 & a^*b\\ b^*a & |b|^2\end{pmatrix} \tag{6}$$

if one chose the basis $|A+\rangle|+\rangle$, $|A-\rangle|-\rangle$ for the combined system.

The appearance of the off-diagonal matrix elements in the density matrix in the basis of 'pointer eigenstates' of the apparatus is the problem of measurement in quantum mechanics. In the presence of these terms, it is wrong to assert that the pointer *is* either in the left half or *is* in the right half. There is another way to see this. The state in (5) can be written as

$$a|A+\rangle|+\rangle+b|A-\rangle|-\rangle=\frac{1}{\sqrt{2}}[(a|+\rangle+b|-\rangle)|A_1\rangle + (a|+\rangle-b|-\rangle)|A_2\rangle], \tag{7}$$

where

$$A_1=\frac{1}{\sqrt{2}}(|A+\rangle+|A-\rangle);\quad A_2=\frac{1}{\sqrt{2}}(|A+\rangle-|A-\rangle). \tag{8}$$

If, therefore, one allows for the reality of the states A_1 and A_2 of the apparatus, the same measurement may be looked upon as a measurement of another set of states, namely, $(a|+\rangle+b|-\rangle)$ and $(a|+\rangle-b|-\rangle)$ of the quantum system. This set could, for appropriate values of a and b, for example for $a=b=\frac{1}{2}$, be eigenstates of an incompatible observable of the system. It would seem, therefore, that anytime before one looked at an apparatus and finds it in $|A+\rangle$ or $|A-\rangle$, the measurement is incomplete. Such a situation is unsatisfactory, because in any reasonable description of measurement, actual reading of the apparatus should be inessential.

One must require, therefore, that in a measurement process, the density matrix undergoes a two-stage reduction:

$$\rho=\begin{pmatrix}|a|^2 & a^*b\\ b^*a & |b|^2\end{pmatrix}\rightarrow\begin{pmatrix}|a|^2 & 0\\ 0 & |b|^2\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\text{ or }\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{9}$$

The last stage corresponds to the actual reading of the apparatus and resembles collapse of a classical probability distribution as a result of acquisition of knowledge about the system. This has nothing to do with quantum mechanics and is irrelevant in the present context. The first stage of reduction is the crux of the problem under discussion. We must require of a measuring apparatus the following: (i) the off-diagonal elements of the density matrix must disappear; (ii) they must do so in the basis of the pointer eigenstates and (iii) the diagonal matrix elements must stay intact in the process.

The problem, therefore, is *What principle in nature selects the pointer eigenstates as*

the preferred basis? This question arises in a more general context then the problem of quantum measurement. Selection of preferred basis is involved in the explanation of 'classical' behaviour of any macroscopic object. For example, for large objects like tables and chairs, nature selects a set of states localized in real space as the preferred set, for large magnets, states with magnetization oriented in definite directions in real space are selected as the preferred set and so on. The often-stated principle: "quantum behaviour reduces to classical behaviour in the limit $\hbar \to 0$" should really be modified to the statement "quantum mechanics allows for classical behaviour in the limit $\hbar \to 0$". One needs an additional principle for selection of preferred basis before classical behaviour can be 'explained'.

2. A proposed solution

This proposal (Zurek 1983) is based on the general consideration that macroscopic systems cannot be considered isolated and the preferred basis is enforced on them by some feature of the interaction between these systems and the environment. The proposed principle could be stated as: "Nature selects as the preferred basis that basis in which the Hamiltonian of the system-environment interaction is diagonal". Another way of stating this would be, for macroscopic systems that set of states is preferred in nature between which the environment does not produce transitions.

3. A simple illustration

In this section I shall try to illustrate the operation of this principle in a very simple system which has been of interest to me for quite some time. This is the sugar molecule which exists in nature as two distinct optically active isomers; the left-handed and right-handed sugar which are mirror-images of each other. They can be experimentally distinguished by the sense of rotation of the plane of plane-polarized light passing through a solution containing one or the other kind of sugar. It is known that if one of the above two kinds of sugar is prepared, it stays in that state almost for ever. Let us call the left-handed and the right-handed states $|L\rangle$ and $|R\rangle$. The sugar molecule poses a well-known paradox. The hamiltonian of the entire molecule is invariant under reflection symmetry, since there are no parity-violating interactions present. The true eigenstates of the molecule should therefore be states which are also eigenstates of the parity operation. The $|L\rangle$ and $|R\rangle$ states are, however, not. For some reason, nature has given up energy eigenstates in favour of the broken-symmetry states, namely the $|L\rangle$ and $|R\rangle$ states as the preferred states.

Let us consider the simple two-state model for this system in which the molecule is approximated to be a system in a symmetric double potential well, the potential being a function of some configurational coordinate that takes a left-handed molecule into a right-handed one and vice-versa. If the potential barrier is infinitely high, there are two degenerate states, one localized at the bottom of each well, corresponding to the $|L\rangle$ and $|R\rangle$ states. When the barrier height is finite, there is an amplitude, say A, for an $|L\rangle$ state to tunnel into $|R\rangle$ per unit time and vice-versa. The hamiltonian matrix in the basis can be written as

$$H = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}. \tag{10}$$

The energy eigenstates of this system, $|+\rangle$ and $|-\rangle$ separated in energy by $2A$ are,

$$|+\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle); \quad |-\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle). \tag{11}$$

The density matrix at time t is given by

$$\rho = \begin{pmatrix} \cos^2\frac{At}{\hbar} & i\cos\frac{At}{\hbar}\sin\frac{At}{\hbar} \\ -i\cos\frac{At}{\hbar}\sin\frac{At}{\hbar} & \sin^2\frac{At}{\hbar} \end{pmatrix} \tag{12}$$

Let us now introduce some interaction of this system with the environment in the form of collisions of random strength, separated by time interval τ and make the crucial assumption that this interaction is diagonal in the $|L\rangle, |R\rangle$ basis. In other words, the collisions do not induce transitions between the $|L\rangle$ and $|R\rangle$ states, i.e., do not affect the diagonal elements of ρ. The only possible effect is, therefore, to introduce a random phase ϕ between the amplitudes of $|L\rangle$ and $|R\rangle$. After the first collision at time τ, therefore

$$\rho(\tau) = \begin{pmatrix} \cos^2\frac{A\tau}{\hbar} & i\langle e^{i\phi}\rangle\cos\frac{A\tau}{\hbar}\sin\frac{A\tau}{\hbar} \\ -i\langle e^{i\phi}\rangle\cos\frac{A\tau}{\hbar}\sin\frac{A\tau}{\hbar} & \sin^2\frac{A\tau}{\hbar} \end{pmatrix} \tag{13}$$

The net effect of the collision is, therefore, to reduce the magnitude of ρ_{12} by an amount depending upon the magnitude of ϕ which can lie anywhere between $-\pi$ and $+\pi$. The exact time evolution of ρ in the presence of collisions is difficult to solve except in a very special case, when ϕ has a uniform distribution between $-\pi$ and $+\pi$, so that $\langle e^{i\phi}\rangle = 0$. In this case, ρ_{12} and ρ_{21} become zero after each collision and it can be easily shown that after N steps of evolution,

$$\rho(N\tau) = \begin{pmatrix} \frac{1}{2}\left(1+\cos^N\frac{2A\tau}{\hbar}\right) & 0 \\ 0 & \frac{1}{2}\left(1-\cos^N\frac{2A\tau}{\hbar}\right) \end{pmatrix}. \tag{14}$$

In the limit of very frequent collisions, i.e., for $2A\tau/\hbar \ll 2\pi$ we have

$$\rho_{11} = 1 - \frac{NA^2\tau^2}{\hbar^2} = 1 - \frac{A^2t}{n\hbar^2}, \tag{15}$$

where t is the total time elapsed and n is the frequency of collision, so that $t = N\tau$ and $n = 1/\tau$.

It follows from (15) that if collisions are frequent enough, ρ_{11} can be made arbitrarily close to 1 for any given time t, i.e., the system can be frozen in state $|L\rangle$ by frequent collisions. Similar effects have been discussed in literature in various contexts under various names, e.g., 'quantum watch-dog effect', 'a watched pot never boils' and so on (Misra and Sudarshan 1977; Joos 1984; Horwitz and Katznelson 1983; Harris and

Stodolsky 1981). The point to note here is that the absence of off-diagonal elements in the system-environment interaction hamiltonian is crucial to the freezing of the $|L\rangle$ state.

An alternative way of looking at the problem makes the above point clear and also helps in studying the evolution of the density matrix for arbitrary strength of the collisions. Let us note that the two-state problem can be mapped exactly onto a spin $-1/2$ problem. Further, all the states of a spin $-1/2$ particle can be mapped onto points on the surface of a sphere in three dimensions. If the axes are chosen as shown in figure 1 with $|L\rangle$ and $|R\rangle$ located at the poles, a point with spherical coordinates (θ, ϕ) on the sphere represents a state

$$\psi = \exp(i\phi/2)\cos\theta/2\,|L\rangle + \exp(-i\phi/2)\sin\frac{\theta}{2}|R\rangle. \tag{16}$$

Also, any arbitrary hamiltonian can be written in appropriate units, as

$$H = -\boldsymbol{\sigma}\cdot\boldsymbol{B}, \tag{17}$$

where $\boldsymbol{B}$ is a magnetic field in some direction and σ_x, σ_y and σ_z are the Pauli matrices. With axes chosen as in the figure, hamiltonian (10) corresponds to a magnetic field of strength A in the x-direction. The quantum evolution of the state point on the surface is a circular orbit (curved dotted line) in the $y-z$ plane with angular frequency $A/\hbar$. The interaction with environment corresponds to a series of magnetic field impulses of random strength in the z-direction. A collision, therefore, rotates the state point about the z-axis by a random angle ϕ depending upon the strength of the collision, leaving θ unchanged (arrowed arc). It may be looked upon as a diffusion in the ϕ-coordinate. With this picture, the problem lends itself easily to a computer simulation, where one starts with a sample of, say, 100 spins at one of the poles, allows them to evolve and

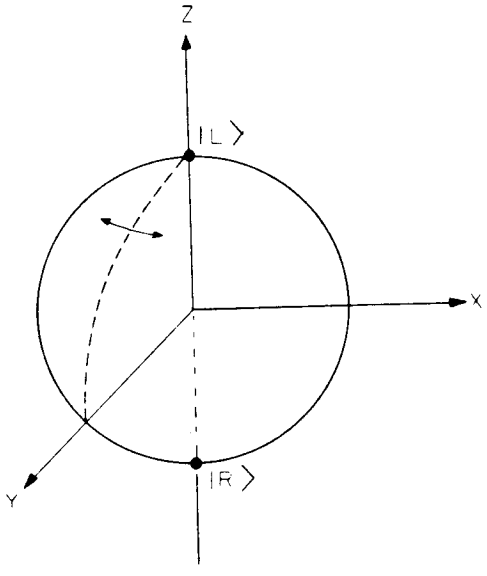

Figure 1. The state sphere and the evolution track for the two-state system.

undergo diffusion and keep track of ρ_{11}, which is given by $\langle \cos^2 \frac{1}{2}\theta \rangle$ over the ensemble. Two parameters are used in the simulation. The random 'kick' in ϕ is taken to be uniformly distributed between $+\phi_0$ and $-\phi_0$, a measure of 'strength' of the collision. The number of collisions per period of the quantum oscillation of the amplitude, N, is the second parameter. The results of the computer simulation are briefly as follows:

(i) When $\phi_0 = \pi$ we recover the analytical result mentioned above.

(ii) In the limit $\phi_0 \sqrt{N} \ll \pi$, ρ_{11} behaves like a weakly damped harmonic oscillator with a decay time $\sim 2\pi^2 \hbar / A\phi_0 \sqrt{N}$.

(iii) For $\phi_0 \sqrt{N} \simeq \pi$ one has the situation analogous to critical damping and

(iv) If ϕ_0 or N is increased further, the decay of ρ_{11} is slowed down. In the limit of large N, ρ_{11} does not decay.

4. Conclusions

The behaviour of the two-state system in the presence of the assumed random interaction with the environment may be summarized as follows: In the absence of interaction, the probability for $|L\rangle$, i.e., ρ_{11} oscillates indefinitely. Thermal equilibrium is, therefore, never attained. For small interaction (weak damping), ρ_{11} decays to $\frac{1}{2}$, hence equilibrium attained with a finite time constant which decreases with increasing strength of the interaction. After a critical strength of the interaction, the decay time increases with increasing strength of the interaction (heavy damping).

It is important to note that instead of $|L\rangle$ or $|R\rangle$ if one started with a state like $|+\rangle$ or $|-\rangle$, i.e., with a point on the equator of the state sphere, this state cannot be frozen as above, no matter how frequent the collisions. Instead, the state would become uniformly distributed along the equator in a finite time; decreasing with the strength and frequency of collisions.

The above is true as long as collisions imitate a magnetic field in the z-direction. If one introduces collisions that take $|L\rangle$ to $|R\rangle$, i.e., off-diagonal elements in the $|L\rangle, |R\rangle$ basis, these imitate a magnetic field in the x–y plane. It can be easily verified that in the presence of such a field, once again $|L\rangle$ and $|R\rangle$ states cannot be frozen. The structure of the system environment interaction is, thus, the crucial factor that stabilizes the $|L\rangle$ and $|R\rangle$ states.

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